A Statistical View of Differential Privacy

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Joint work with Shuheng Zhou
Ongoing work with Shuheng Zhou and Alessandro Rinaldo
Overview


II  Statistical Thinking.

III  Sanitized databases.

IV  Problems with differential privacy.
I: A Short Story

- I live in several worlds:
  - statistical theory
  - applications (astronomy, biology, genomics)
  - machine learning (theory and methodology)
- Steve Fienberg got me and a post-doc (Shuheng Zhou) interested in privacy. I dabbled a bit and we wrote a paper.
- I was invited to IPAM: there was statisticians and computer scientists.
- The statisticians were mostly applied statisticians working on real problems.
- The CS people were mainly theoreticians doing very interesting theory.
A Short Story

• There was a huge cultural divide.

• The CS people wanted precise definitions of privacy and theorems guaranteeing that privacy (mostly differential privacy) held. I liked that.

• The statisticians wanted methods that worked on real, complex, data sets. I liked that.

• With a few notable exceptions (Steve, Adam, Cynthia, ....) they ignored each other.
A Short Story

• CS view: receive a query, return a private answer.

• Statistics view: give me data. Then I can: draw plots, fit models, test fit, estimate parameters, make predictions, ...

• The moral of the story: statisticians want a sanitized database, not answers to specific queries.

• I have a dream: statisticians will read the CS literature and CS people will read the statistics literature. There is a huge opportunity for collaboration.

• You will notice many unfinished items marked: in progress.
Overview


II Statistical Thinking.

III Sanitized databases.

IV Problems with differential privacy.
Statistical Thinking
or Some Statistical Concepts

- What do statisticians do?
  - Exploratory methods
  - Model fitting
  - Looking at residuals
  - Assessing fit
  - Develop new methods
  - Theory

- We view these as very inter-connected.
Some Statistical Concepts

• Data $D = (X_1, \ldots, X_n)$ where $X_1, \ldots, X_n \sim P$.

• Often (but not always) we view the data base as a sample from a population. The goal is not just to summarize the database; we want to infer (learn) about the population.

• Formally, the goal is to infer $P$ or some functions of $P$ (means, correlations, etc.) or predict a new observation.

• A model is a set of distributions $\mathcal{P}$.

• Can be parametric: Example: $\mathcal{P} =$ Normal distributions.

• Can be nonparametric: Example: $\mathcal{P} =$ all distributions.
Point estimation

- Let $\theta = T(P)$. (Example, $T(P)$ is the mean of $P$.)

- Estimator: $\hat{\theta} = g(X_1, \ldots, X_n)$.

- Loss function $\ell(\hat{\theta}, \theta)$. Example: $\ell(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$.

- Optimality: find $\hat{\theta}$ that achieves the minimax risk:

$$R_n = \inf_{\hat{\theta}} \sup_{P \in \mathcal{P}} \mathbb{E}_P[\ell(\hat{\theta}, \theta)]$$

- How is $R_n$ affected by privacy?
• **Example:** If \( \theta \) is the mean and \( \mathcal{P} \) is all Normals, then

\[
\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i
\]

is minimax for all bowl-shaped loss functions.

• \( R_n = O(1/\sqrt{n}) \). Extra loss from differential privacy is \( O(1/n) \).

\[
\frac{|R_n - R^*_n|}{R_n} = 1 + O \left( \frac{1}{\sqrt{n}} \right).
\]

* This is not quite true. The real statement requires working on a bounded domain and involved a few complications.
• **Example**: Estimating a probability density function.

• Observe from $X_1, \ldots, X_n \sim p$, where $X_i \in \mathbb{R}^d$. Estimate $p$.

• Loss $\int (p(x) - \hat{p}(x))^2 dx$ where $\hat{p}$ is the estimator.

• Minimax risk:

$$\inf_{\hat{p}} \sup_{p \in \mathcal{P}} \mathbb{E}_p \int (p(x) - \hat{p}(x))^2 dx = \frac{C}{n^{4/(4+d)}}$$

where $\mathcal{P}$ is all smooth densities.

• We have several methods for estimating $p$ that are optimal. This will be useful when we discuss sanitized databases. More later.
Confidence Intervals

• Possibly the most important and most common statistical calculation.

• Find a (random) set $C = C(X_1, \ldots, X_n)$ such that

$$P(\theta \in C) \geq 1 - \alpha \quad \text{for all } P.$$

• Again, can be parameteric or nonparametric.

• Optimality: want $C$ to be as small as possible.

• Example: correlation between a SNP and a disease.
Prediction

• Data \((X_1, Y_1), \ldots, (X_n, Y_n)\).

• Observe new \(X_{n+1}\). Predict \(Y_{n+1}\).

• If \(Y \in \mathbb{R}\) this is regression. If \(Y\) is discrete this is classification.

• Classification is a small (but important) part of statistics.
Other Things Statisticians Do

- Clustering.
- PCA.
- Curve Estimation.
- Experimental design.
- Graphical models.
- Networks.
- Manifold methods.
- Hypothesis testing.
- Yada yada yada.
Overview


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How To Make a Sanitized Database
(And How To Measure Accuracy)

- Data $X = (X_1, \ldots, X_n)$. $X \in \mathcal{X} =$ set of possible databases.

- $P \xrightarrow{} X = (X_1, \ldots, X_n) \xrightarrow{} Q \xrightarrow{} Z = (Z_1, \ldots, Z_k)$

where $Q(Z_1, \ldots, Z_k | X_1, \ldots, X_n)$ generates random (synthetic) data.

- Require:

$$Q(Z \in B | X) \leq e^\alpha Q(Z \in B | X')$$ for all $B$

whenever $X \sim X'$ (differ by one observation).
Method I: Density Estimation.

(1) Choose a basis: $\psi_1, \psi_2, \ldots$, (Fourier, polynomial, wavelets, ....)

Expand: $p(x) = \sum_j \beta_j \psi_j(x)$ where $\beta_j = \int \psi_j(x)p(x)dx$.

(2) Estimate $p$ with

$$\hat{p}(x) = \sum_{j=1}^{J} \hat{\beta}_j \psi_j(x)$$

where

$$\hat{\beta}_j = \frac{1}{n} \sum_{i=1}^{n} \psi_j(X_i).$$
Method I: Density Estimation.

(3) Privatize:

\[ \tilde{\beta}^*_j = \tilde{\beta}_j + \frac{cJ}{n} L_j \]

where \( L_1, \ldots, L_J \) are Laplace random variables. Let

\[ \tilde{p}^*(x) = \sum_{j=1}^{J} \tilde{\beta}^*_j \psi_j(x). \]

(4) Sample from \( p^* \):

\[ Z_1, \ldots, Z_k \sim \tilde{p}^*. \]
• Histograms:

• Add Laplace noise to the counts, sample from the histogram. (This is less accurate.)
• Kernel density estimator:

\[
\hat{p}(x) = \frac{1}{n} \sum_{i=1}^{k} \frac{1}{h^d} K \left( \frac{||x - X_i||}{h} \right)
\]

where \( h > 0 \) is a bandwidth and \( K(\cdot) \) is a kernel.

• Privatize (sample and aggregate) then sample from \( \hat{p}^* \). (in progress).
Technical point:

Each density estimate requires careful choice of tuning parameter:

- series estimator: number of terms
- histogram: size of the bins
- kernel estimator: bandwidth

We do this using risk estimation.

This needs to be privatized too.

In progress.
Method II: Exponential Mechanism (McSherry and Talwar 2007)

• First need to explain: (i) empirical distributions and (ii) metrics on distributions.

• Let $P_X$ be the distribution that puts mass $1/n$ at each $X_i$. 
Some Metrics \( d \)

- **Supremum distance:** \( d(P, Q) = \sup_{A \in \mathcal{A}} |P(A) - P(A)|. \)

- **Wasserstein: (Earth-mover):**
  \[
  d(P, Q) = \inf_{J} \mathbb{E}_J \|X - Z\|^p
  \]
  where \( X \sim P, Y \sim Q \) and the infimum is over all joint distributions \( J \) with marginals \( P \) and \( Q \).

- **\( L_2 \) distance on counts:** \( \|x - z\| = \sqrt{\sum_j (x_j - z_j)^2} \).

- **\( L_1 \) distance on densities:** \( \int |p - q| \).
Exponential mechanism

- Draw $Z = (Z_1, \ldots, Z_k)$ from
  \[
  q(z_1, \ldots, z_k|x_1, \ldots, x_n) \propto \exp\left(-\frac{\alpha d(P_x, P_Z)}{2\Delta}\right)
  \]
  where
  \[
  \Delta = \Delta(n, k) = \sup_{x,x'} \sup_z |d(P_x, P_z) - d(P_{x'}, P_z)|.
  \]

- Can do the sampling by importance sampling but it is difficult.
ACCURACY

Let $Q_\alpha$ be all $Q$'s that satisfy $\alpha$-differential privacy.

$$R(\mathcal{X}) = \inf_{Q \in Q_{\alpha}} \sup_{x \in \mathcal{X}} \mathbb{E}_Q(d(P_x, P_Z))$$

$$R(\mathcal{P}) = \inf_{Q \in Q_{\alpha}} \sup_{P \in \mathcal{P}} \mathbb{E}_P \mathbb{E}_Q(d(P_X, P_Z)).$$

This has been in a few cases. (See Hardt and Talwar 2007, Roth 2010, and Kasiviswanathan, Rudelson, Smith and Ullman 2010 for example.) Mostly, it is work in progress.

Less ambitious is to compute:

$$R(\mathcal{P}, Q) = \sup_{P \in \mathcal{P}} \mathbb{E}_P \mathbb{E}_Q(d(P_X, P_Z))$$

for some specific $Q$'s.
An Accuracy Bound

- Draw $Z = (Z_1, \ldots, Z_k)$ from
  $$q(z|x) \propto e^{-\alpha d(P_x, P_z)/(2\Delta)}$$

where

$$\Delta = \Delta(n, k) = \sup_{x, x'} \sup_z |d(P_x, P_z) - d(P_{x'}, P_z)|.$$ 

- Theorem (WZ 2010):
  $$\mathbb{P}(d(P, P_Z) > \epsilon) \leq \frac{(\sup_x p(x))^k e^{-3\alpha \epsilon/(16\Delta)}}{R(k, \epsilon/2)}$$

where

$$R(k, \epsilon) = \mathbb{P}(d(P, P_S) \leq \epsilon) \quad S = (S_1, \ldots, S_k) \sim P$$

is the small ball probability.
### Dimension $r$

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<th>Distance</th>
<th>Data Release mechanism</th>
<th>minimax rate</th>
</tr>
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<tr>
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<td>perturbed histogram</td>
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<td></td>
<td>exponential mechanism</td>
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<tr>
<td>$L_2$</td>
<td>$n^{-2/(2r+3)}$</td>
<td>$n^{-2/(2+\gamma)}$</td>
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<td>$\sqrt{\log n} \times n^{-2/(6+r)}$</td>
<td>$\log n \times n^{-2/(2+r)}$</td>
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<td></td>
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<tr>
<td></td>
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<td>$n^{-1/3}$</td>
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### Dimension $r = 1$

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<tbody>
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<td>$n^{-\gamma/(2\gamma+1)}$</td>
<td>$n^{-2\gamma/(2\gamma+1)}$</td>
<td>$n^{-2\gamma/(2\gamma+1)}$</td>
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</table>

**Main point:** general picture on optimality not yet clear.
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Problems With Differential Privacy

• Differential privacy is a precise and strong guarantee.

• But there are two problems:

  • First, recall that $\mathcal{X} =$ set of possible databases. What is $\mathcal{X}$? It is ambiguous.

  • And as Avrim pointed out, the notion of neighboring databases can be ambiguous.

  • Also, $\mathcal{X}$ can be exotic. For example, in functional data analysis, each data point is a function living in some function space.
Problems With Differential Privacy

- Consider histogram counts: \( X = (c_1, \ldots, c_k) \) where \( \sum_j c_j = n \).

- Lower bounds:
  Hardt and Talwar (2009): \( O(k^{3/2}) \). We got: \( O(k^{3/2}/n) \).

- Depends on whether \( \mathcal{X} \) is all histograms or \( \mathcal{X} \) histograms with sample size \( n \).

- In many real problems, it simply is not clear what \( \mathcal{X} \) is.

- Need to know \( \mathcal{X} \) to even implement differential privacy.
Problems With Differential Privacy

- Second, it is too strong.

- Consider a high dimensional contingency table. The counts are very sparse. There are many zeroes.

- The sample size is \( n \) is much smaller than the number of cells.

- Creating a synthetic database subject to differential privacy leads to a very noisy database. (Mostly noise.)
CONCLUSION

• Differential privacy is a precise, mathematical guarantee.

• This precision is useful theoretically but makes it somewhat impractical.

• Popular in CS. Mostly ignored in statistics.

• Need modified version of differential privacy?