Sum of Squared Edges for MST of a Point Set in a Unit Square

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1 Introduction

Let the *weight* of a tree be the sum of the squares of its edge lengths. Given a set of points P in the unit square let $W(P)$ be the weight of the minimum spanning tree of P, where an edge length is the Euclidean distance between its endpoints. If P consists of the four corners of the square, then $W(P) = 3$. Gilbert and Pollack [2] proved that $W(P)$ is $O(1)$ and this was extended to an arbitrary number of dimensions by Bern and Eppstein [1]. While more recent divide-and-conquer approaches have shown that $W(P) \leq 4$, no point set is known with $W(P) > 3$, and hence it has been widely conjectured (e.g. see [4]) that $W(P) \leq 3$. Here we show that $W(P) < 3.41$.

For a point set P in a unit square, $MST(P)$ denotes a minimum spanning tree of P. Let $MST_k(P)$ denote the subgraph of $MST(P)$ in which all edges of length greater than k have been removed from $MST(P)$. For any given point $X \in P$, define $MST_k(X, P)$ to be the connected component of $MST_k(P)$ containing X. Let $\boxplus$ be the set of corners of the unit square.

Lemma 1. *For all P, $W(P) \leq W(P \cup \boxplus)$.*

Lemma 2. *No edge in $MST(P \cup \boxplus)$ has length greater than 1.*

2 Bounding the Weight of the MST

By Lemma 1 it suffices to consider only point sets that include the corners of an enclosing unit square.

Kruskal's MST construction algorithm [3] considers all possible edges defined by P in order of increasing length. When an edge is considered, it is added to the existing graph only if no cycle is created. Let e_m be the m^{th} edge added. At step $m = 0$ no edges have been added and at step $m = |P| - 1 =: M$, $MST(P)$ is complete.

At each step of Kruskal's algorithm, each connected component is a tree. We define t_X^m to be the tree at step m that contains point X. It helps to initialize the algorithm at $m = 0$ by letting every point $X \in P$ be a singular tree t_X^0 that is augmented when an edge is added between X and some other point of P. We also initialize $|e_0|$ to be 0. Notice that $t_X^m = MST_{|e_m|}(X, P)$. Let $\mathcal{CH}(t)$ denote the vertices of the convex hull of a tree t. If X is on $\mathcal{CH}(t_X^m)$, let $\angle^m(X)$ be the range of angles for which X is extreme with respect to the vertices of t_X^m. We set $\angle^0(X) = [0°, 360°]$. Over time this range of angles is reduced, and may have size 0 if X is no longer in the convex hull. At any time m, for any given connected component Z, the set of all $\angle^m(X)$ for each point $X \in Z$ partitions the angle range [0, 360].

With this in place we define the region $C^m(X)$ as follows. At time m, if X is on $\mathcal{CH}(t_X^m)$ and extreme in some range $[\alpha, \beta] = \angle^m(X)$, then $C^m(X)$ is the sector of a circle centered at X, with radius $\frac{|e_m|}{2}$ and spanning the angle range $[\alpha, \beta]$. If X is not in $\mathcal{CH}(t_X^m)$, then $C^m(X)$ is empty. Let $C^{*m}(X)$ be the union of all the sectors that X has defined up to step m; that is $C^{*m}(X) = \cup_{\mu=0}^{m} C^{\mu}(X)$.

For a tree t_X^m, we define the region A_X^m as follows.

$$A_X^m = \bigcup_{Y \in t_X^m} C^{*m}(Y).$$

Notice that A_X^m is contained in the union of discs of radius $\frac{|e_m|}{2}$ centered on all points of P in the same component as X. Points in separate components have distance greater than $|e_m|$, otherwise an edge between them would have already been added and they would be in the same component. Thus if $A_X^m \neq A_Y^m$, the two regions are disjoint. Let A^m be the union of A_i^m for all $i \in P$ and let Φ^m be the area of A^m at time m.

At time m, there are $|P| - m$ trees. (Points of P not yet joined to other points are also considered to be trees.) Let $\ell^m = |e_m|^2$.

Lemma 3. $\Phi^{m+1} = \Phi^m + \frac{\pi}{4}(|P| - m)(\ell^{m+1} - \ell^m)$.

Proof. At time m, each point X in $\mathcal{CH}(t_X^m)$ has a sector $C^m(X)$ with radius $\frac{|e_m|}{2} = \frac{\sqrt{\ell^m}}{2}$. From our definition of $C^m(X)$, the sectors of all points in $\mathcal{CH}(t_X^m)$ partition a circle of radius $\frac{\sqrt{\ell^m}}{2}$, which has area $\frac{\pi \ell^m}{4}$. From step m to $m+1$, the radius of

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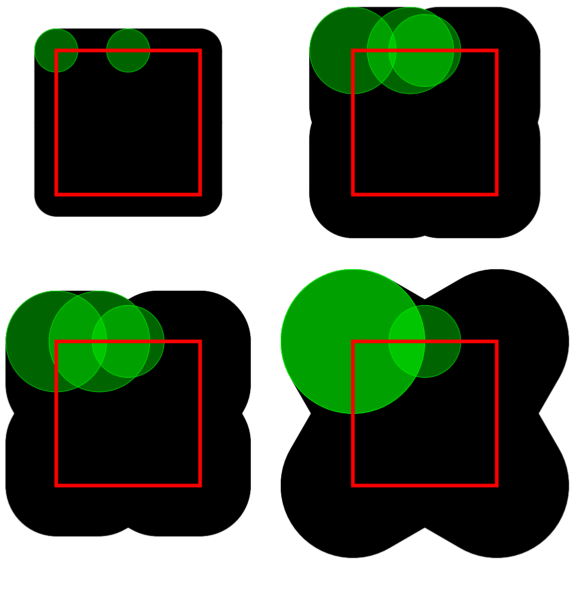

Figure 1: Depiction of R for $d \in \{0.3, 0.6, 0.7, 1.0\}$.

each of these sectors increases to $\frac{\sqrt{\ell^{m+1}}}{2}$ and the total area of the partitioned circle increases to $\frac{\pi \ell^{m+1}}{4}$. There are $|P|-m$ trees that each have this growth, and whose regions are disjoint, so multiplying the difference $\frac{\pi}{4}(\ell^{m+1}-\ell^m)$ from each tree by the number of trees, $|P|-m$, gives the result. □

Let W^m denote the sum of the weights of all trees at step m.

Lemma 4. $W^m = \frac{4}{\pi}\Phi^m - (|P|-m)\ell^m$.

Proof. We induct on m. For the base case, $m = 0$, the spanning tree consists of no edges and the points are disconnected. Consequently, $W^0 = 0$, $\Phi^0 = 0$, and $\ell^0 = 0$. Assume that the statement holds for W^m. We will prove that it holds for W^{m+1}.

Because e_{m+1} is the edge added at step $m+1$, $W^{m+1} = W^m + \ell^{m+1}$. We substitute W^m with the value from our induction hypothesis to obtain $\frac{4}{\pi}\Phi^m - (|P|-m)\ell^m + \ell^{m+1}$. By substitution, using Lemma 3, this equals $\frac{4}{\pi}\Phi^{m+1} - (|P|-m)\ell^{m+1} + \ell^{m+1}$. Simple algebra yields the claimed result. □

Lemma 5. *Let d denote $|e_M|$. If $d \le \frac{1}{2}$, $\Phi^M \le 2d + \frac{\pi d^2}{4} + 1$. Otherwise, $\Phi^M \le d^2\sqrt{3} - \frac{1}{\sqrt{3}} + \frac{5\pi d^2}{12} + 4(d-d^2)+1$.*

Proof. In Figure 1, we depict a region R that we claim covers A^M. For every point x on each edge $\overline{e}$ of the square, define a circle of radius $\min\{\frac{d}{2}, \frac{f}{2}\}$, where f is the distance from x to the farther endpoint of $\overline{e}$. This circle is meant to cover the area that could be occupied by the region of a point at x. If x were a point in P and its region exceeded this circle outside the unit square, then its region would intersect both growing circles centered on the endpoints of $\overline{e}$. Therefore x would be connected to both endpoints and it would no longer be extremal in the direction outside of the square. This would further imply that no sector of x could keep expanding outside the square once $C^*(x)$ exceeds this circle. We define R to be the union of all such circles centered on the boundary of the square, together with the square region itself. This represents an upper bound on the region that A^M can occupy, as the extreme case occurs when points in P are located on the boundary of the square.

It remains to show that R cannot grow any more due to points of P inside the square. Suppose that an interior point y grows some sector $C^m(y)$ that contributes towards Φ^M outside R. Without loss of generality let this extra contribution be closest to the top edge $\overline{e}$ of the square. Just like above, $C^m(y)$ can only grow above $\overline{e}$ if y is part of the upper hull of t_y^m and that cannot happen if y is in the same component as both endpoints of $\overline{e}$. Let x be the orthogonal projection of y on e and assume without loss of generality that the endpoint of $\overline{e}$ farthest from y is the right endpoint r. Therefore the endpoint of $\overline{e}$ farthest to x is also r. Furthermore, the midpoints of $\overline{xr}$ and $\overline{yr}$ have the same x-coordinate. Therefore, the portion of $C^M(y)$ above $\overline{e}$ is contained in the circle of radius $\min\{\frac{d}{2}, \frac{f}{2}\}$ centered at x, which contradicts the assumption. All that remains is to calculate the area of R. This can be done algebraically but details are omitted from this version. □

Theorem 6. *For any set of points P in the unit square, $W(P) \le \frac{3\sqrt{3}+4}{\pi} - \frac{1}{\pi\sqrt{3}} + \frac{2}{3} \approx 3.4101$.*

Proof. From Lemma 1, we can assume that P includes the corners of its enclosing unit square. $W^M = W(P)$, and by Lemma 4 is equal to $\frac{4}{\pi}\Phi^M - \ell^M$. This in turn is bounded in terms of d in Lemma 5. Combining, we obtain the following upper bounds on $W(P)$ in terms of d: $W(P) \le \frac{4d^2\sqrt{3}+16(d-d^2)+4}{\pi} - \frac{4}{\pi\sqrt{3}} + \frac{5d^2}{3} - d^2$ when $d > 0.5$ and $W(P) \le \frac{8d+4}{\pi}$ for $d \le 0.5$. This function is monotonically increasing for $0 \le d \le 1$, so substituting $d = 1$ and simplifying gives the claimed upper bound of $\frac{4\sqrt{3}+4}{\pi} - \frac{4}{\pi\sqrt{3}} + \frac{5}{3} - 1 \approx 3.4101$. □

References

[1] M. W. Bern and D. Eppstein. Worst-case bounds for subadditive geometric graphs. In *Proc. 9th Symp. on Comp. Geom.*, pages 183–188, 1993.

[2] E. N. Gilbert and H. O. Pollack. Steiner minimal trees. *SIAM J. App. Math.*, 16(1):1–29, 1968.

[3] J. B. Kruskal. On the shortest spanning subtree of a graph and the traveling salesman problem. *Proc. American Math. Soc.*, 7(1):48–50, Feb 1956.

[4] D. B. West. `http://www.math.uiuc.edu/~west/regs/mstsq.html`.