ON THE INTERPRETATION OF SCOTT'S DOMAINS †

John C. Reynolds
Syracuse University

Abstract The lattice-theoretic approach to the theory of computation, developed by D. Scott, is based on assumptions that the data spaces manipulated by a computation are complete lattices whose partial ordering represents a notion of approximation, and that the computable functions between such data spaces are continuous (in an appropriate sense). Since these assumptions lead to conclusions which are quite different from the conventional theory of computation, it is important to understand their interpretation in terms of actual computational processes.

We consider non-terminating computational processes whose input and output are enumerations of sets of messages. Given a relationship of satisfaction between the universe of messages and some universe of models, we define the meaning of a message set to be the set of those models which satisfy all of its members, and we define the domain to be the range of this meaning function. In this interpretation, Scott's axioms can be justified by physical limitations of the communication between the computational processes and their environments.

† Work supported by Rome Air Force Development Center Contract No. 30602-72-C-0281 and ARPA Contract No. DAHC04-72-C-0003.

This is a preprint of a paper to be given at the Convegno d'Informatica Teorica, Istituto Nazionale di Alta Matematica (Citta Universitaria), Rome, Italy, February 6, 1973.

Introduction

The last few years have seen the development of a radical new approach to the theory of computation which is based on assumptions that the data spaces manipulated by a computation are complete lattices whose partial ordering represents a notion of approximation, and that the computable functions between such data spaces are continuous (in an appropriate sense). Originally developed by D. Scott, this approach has been applied to the definition of higher-order programming languages and to the problem of proving program correctness by Scott, Strachey, Wadsworth, Milner, and others.

Invariably, the theory has been presented as an abstract mathematical development from assumptions which are only informally and intuitively justified. However, since these assumptions lead to conclusions which are quite different from the conventional theory of computation, it is important to understand their precise interpretation in terms of real computation.

In this paper, we will explore a particular interpretation of Scott's theory in terms of concrete computational processes, and we will try to justify the axioms of the theory in terms of reasonable assumptions about the physical limitations of such processes. It should be emphasized at the outset that our interpretation is not intended to be exclusive. By itself, Scott's theory is good mathematics - a collection of surprisingly strong inferences drawn from surprisingly weak assumptions. As such, it is likely to have a long and varied history of interpretation and application.

Fundamentally, our interpretation is based on two premises:

(1) The theory is capable of describing useful non-terminating computational processes which accept and produce endless sequences of information.

(2) The axioms of the theory reflect limitations on the communication between such processes and their environment, rather than limitations on the internal character of the processes themselves. In effect, we are dealing with a theory of communication rather than a theory of computation.

More specifically, we consider processes (either mechanical or human) which communicate with their environment via discrete, one-way channels of finite capacity. Each channel transmits a possibly endless sequence of <u>messages</u> selected from a countable universe $\mathcal{M}$ of messages which is characteristic of the channel.

Even at this stage, it is evident that endless communication can only be meaningful in the presence of some notion of approximation and limit. Since the receiver of an endless sequence of messages is never aware of more than a finite initial subsequence, any meaning conveyed by the endless sequence must be a limit of the meanings of its initial subsequences. Moreover, the meaning of a finite sequence must approximate (i.e., be compatible with) the meanings of all of its possible extensions.

Summary of Definitions and Axioms

Before proceeding further, we give a brief summary of the main definitions and axioms used in the lattice-theoretic approach. More detailed expositions are given in several of the references.

A partially ordered set is a complete lattice iff every subset possesses a least upper bound. The symbols D, $\sqsubseteq$, and $\bigsqcup$ (with occassional decoration) will always denote a complete lattice, its partial ordering, and its least upper bound.

A set $X \subseteq D$ is called directed iff every finite subset of X is bounded by some member of X.

A function f from D to D' is said to be:

(1) monotonic, iff $x \sqsubseteq y$ implies $f(x) \sqsubseteq f(y)$ for all $x, y \in D$.

(2) continuous, iff $f(\bigsqcup X) = \bigsqcup\{ f(x) \mid x \in X \}$ for all directed $X \subseteq D$.

(3) additive, iff $f(\bigsqcup X) = \bigsqcup\{ f(x) \mid x \in X \}$ for all $X \subseteq D$.

By taking $X = \{x, y\}$, which is directed when $x \sqsubseteq y$, it is easy to see that continuous functions must be monotonic. We write $D \Rightarrow D'$ $(D \rightarrow D')$ for the set of all functions (all continuous functions) from D to D'.

A set $U \subseteq D$ is called open iff (1) For all $x \in U$ and $y \in D$, $x \sqsubseteq y$ implies $y \in U$, and (2) For all directed $X \subseteq D$, if $\bigsqcup X \in U$, then $x \in U$ for some $x \in X$. It can be shown that unions and finite intersections of open sets are open sets, and that a function $f \in D \Rightarrow D'$ is continuous if and only if, for all open $U' \subseteq D'$, the preimage $\{x \mid x \in D \text{ and } f(x) \in U'\}$ is open.

For x, z ε D, we write $x \prec z$ iff there is an open set $U \subseteq D$ such that $z \varepsilon U$, and $x \sqsubseteq y$ for all $y \varepsilon U$. D is called a continuous lattice iff $y = \bigsqcup \{ x \mid x \varepsilon D \text{ and } x \prec y \}$ for all $y \varepsilon D$.

A set $E \subseteq D$ is called a subbasis of D if $x = \bigsqcup \{ e \mid e \varepsilon E \text{ and } e \sqsubseteq x \}$ for all $x \varepsilon D$. If, in addition, the least upper bound of every finite subset of E is a member of E, then E is called a basis of D. D is said to be countably based if it possesses a countable basis, or equivalently, if it possesses a countable subbasis.

These definitions permit a concise statement of the basic assumptions underlying the lattice-theoretic approach:

A domain is a complete, continuous, countably based lattice.

Meaningful functions between domains are continuous functions.

From these assumptions, Scott has deduced some surprising and provocative conclusions. In particular:

(1) If D and D' are domains, then $D \rightarrow D'$ is a domain under the partial ordering $f \sqsubseteq g$ iff $f(x) \sqsubseteq g(x)$ for all $x \varepsilon D$.

(2) There exists a domain D, with more than one element, which is isomorphic to $D \rightarrow D$.

But the application of such conclusions depends upon our present task: to interpret their underlying assumptions in terms of concrete computational processes.

(In the sequel, we will use several propositions which are minor consequences of the above definitions. Since these propositions are neither deep nor particularly original, and since their proofs would distract from the main thread of our exposition, they have been collected into an appendix.)

The Powerset Interpretation

We first consider an obvious but very special interpretation, in which the meaning of a sequence of messages is simply the set of messages enumerated by that sequence. Since the meaning of a sequence must approximate the meaning of all of its extensions, the relation of approximation must be that of set inclusion.

The set of all meanings is the powerset $2^{\mathcal{M}}$, which is well-known to form a complete lattice under set inclusion. In fact, it is easily shown that $2^{\mathcal{M}}$ is a domain in which, for all $M, N \in 2^{\mathcal{M}}$, $X \subseteq 2^{\mathcal{M}}$:

(1) $M \sqsubseteq N$ iff $M \subseteq N$

(2) $\bigsqcup X = \bigcup X$

(3) $M \prec N$ iff M is a finite subset of N.

(4) The set of all singleton subsets (finite subsets) of $\mathcal{M}$ is a countable subbasis (basis).

This interpretation provides an obvious notion of computing a function between domains. Consider a process P with a single input channel whose message universe is $\mathcal{M}$ and a single output channel whose message universe is $\mathcal{M}'$. We say that P computes the function $F \in 2^{\mathcal{M}} \Rightarrow 2^{\mathcal{M}'}$ iff, for all input sequences which enumerate the set M, P produces an output sequence which enumerates the set F(M).

Now suppose that P computes F, and that $m' \in F(M)$ for some $M \in 2^{\mathcal{M}}$. If an enumeration of M is given to P as its input, then P must emit the message m' at some finite time during its operation. Since the capacity of the input channel is finite, at the time when m' is emitted P cannot have received the entire enumeration of M, but only an enumeration of some finite subset $M_f \prec M$. Thus for any set N containing M_f, we must have $m' \in F(N)$, since when P emits m' it has no way of knowing that its input will not be an enumeration of N.

In particular, this argument shows that $a' \in F(N)$ whenever $M \subseteq N$ (regardless of the value of M_f), and since this holds for all $a' \in F(M)$, we have $F(M) \subseteq F(N)$, i.e. F is monotonic. More interestingly, by taking $N = M_f$, we find that every member of F(M) must belong to the set

$$\bigcup \{ F(M_f) \mid M_f \prec M \}$$

and since the opposite inclusion is an immediate consequence of monotonicity, we have

$$F(M) = \bigcup \{ F(M_f) \mid M_f \prec M \}$$

i.e., F must be finitely generated. By Proposition 3, this is equivalent to the requirement that F be continuous.

It should be noticed that the above argument is based on the nature of the communication between P and its environment, but it does not make any assumptions about the internal nature of P itself (beyond assuming that P cannot predict the future). Thus cuntinuity is more general than the conventional concept of computability. For example, constant functions which produce (conventionally) nonenumerable sets are still continuous.

The fact that Scott's assumptions are weaker than the usual notion of computability may be a considerable virtue. The generalization from computable to continuous functions is much like the generalization from algebraic to real numbers. In both cases one moves from a small but subtle set, determined by a certain kind of finite, implicit representation, to a larger but structurally simpler set which can be constructed by limiting processes.

Logical Interpretations

Since powersets are a limited and special kind of domain, we now develop a more flexible interpretation which can deal with arbitrary domains. We will retain our assumption that a sequence of messages is an enumeration of a set, but we wish to permit the meaning of such a set to be something beyond itself.

For this purpose, we borrow an old idea from mathematical logic: We assume the existence of a universe $\mathcal{U}$ of models and a relation $\sigma \subseteq \mathcal{M} \times \mathcal{U}$ of satisfaction between messages and models. Then the meaning of a set M of messages is the set

$$\alpha(M) = \{ u \mid u \in \mathcal{U} \text{ and for all } m \in M,\ m \,\sigma\, u \}$$

of those models which satisfy every message in M. Intuitively, one can imagine the receiver of a message sequence starting out with the set $\mathcal{U}$ and, upon receipt of each message, pruning out all models which do not satisfy the message. It is tempting to call $\mathcal{U}$ the mental set of the receiver.

Since receiving messages causes the elimination of models, the notion of approximation among model sets must be the inverse of set inclusion. Let $\mathcal{Z}^{\mathcal{U}}$ be the powerset of $\mathcal{U}$ with $\sqsubseteq \; = \; \supseteq$. Then $\mathcal{Z}^{\mathcal{U}}$ is almost a domain, except that it will not be countably based when $\mathcal{U}$ is not countable. Moreover, α is an additive function from $2^{\mathcal{M}}$ to $\mathcal{Z}^{\mathcal{U}}$.

However, rather than using the entire powerset $\mathcal{Z}^{\mathcal{U}}$, we take the domain to be the image of $2^{\mathcal{M}}$ under α:

$$D = \{ U \mid U \in \mathcal{Z}^{\mathcal{U}} \text{ and } U = \alpha(M) \text{ for some } M \in 2^{\mathcal{M}} \}$$

In effect, we are limiting the domain to those meanings which can actually be communicated by message sets. Or from a different viewpoint, we are taking the domain to be the partition of $2^{\mathcal{M}}$ into equivalence classes of message sets with the same meaning.

By Proposition 4, D is a complete lattice with the same partial ordering as $2^{\mathcal{U}}$. Moreover, the image under α of the set of singleton (finite) subsets of $\mathcal{M}$ is a subbasis (basis) of D, so that D is countably based. (However, our construction does not require D to be a continuous lattice; we will return to this point in a moment.) From now on, we will restrict the range of α to D, so that this function continues to be additive and becomes a surjection.

Again there is an obvious notion of computing a function between domains. Suppose P is a process whose input and output channels have message universes $\mathcal{M}$ and $\mathcal{M}'$ which are mapped into meanings by the additive surjections $\alpha \in 2^{\mathcal{M}} \to D$ and $\alpha' \in 2^{\mathcal{M}'} \to D'$. We say that P computes the function $f \in D \Rightarrow D'$ iff, for all input sequences whose meaning is x, P produces an output sequence whose meaning is f(x). This is equivalent to requiring that P compute a powerset function $F \in 2^{\mathcal{M}} \to 2^{\mathcal{M}'}$ such that $\alpha' \cdot F = f \cdot \alpha$.

But if F (and therefore $\alpha' \cdot F$) is continuous and α is additive and surjective, then by Proposition 5, f must be continuous. Thus our argument that computable functions must be continuous extends from the powerset case to the more general logical interpretations.

We are left with the problem of justifying the axiom that domains must be continuous lattices. This is certainly the least intuitive of Scott's assumptions, yet it is critical to the coherence of the theory. For example, if we only require domains to be complete, countably based lattices, then there is a lattice $D \to D'$ of continuous functions which is not countably based.

Admittedly, our interpretation does not seem to make lattice continuity inevitable. But at least there is a natural assumption which is equivalent to lattice continuity:

Since α can assign the same domain element to several message sets, it is natural to look for canonical message sets, i.e., to seek a function $\beta \in D \Rightarrow 2^{\mathcal{M}}$ such that $\alpha \cdot \beta$ is the identity function on D. The existence of such a function is guaranteed by the surjectiveness of α. But suppose we insist that the process of transforming any message set into a canonical message set with the same meaning must be a computable process, so that β must be a continuous function (and β,α must be what Scott calls a retraction pair).

By Propositions 7 and 8, there exists a continuous function β such that $\alpha \cdot \beta$ is the identity on D if and only if D is a continuous lattice. Moreover, if any function meets these requirements, then they are met by the function

$$\beta(x) = \bigsqcup \{ M \mid M \in 2^{\mathcal{M}} \text{ and } \alpha(M) \prec x \}.$$

It should be noticed that this function is different than one might expect from conventional logic. An obvious choice of a canonical message set representing the model set $x \in D$ is the set of all messages which are satisfied by every model in x; this would give

$$\beta(x) = \bigsqcup \{ M \mid M \in 2^{\mathcal{M}} \text{ and } \alpha(M) \sqsubseteq x \}.$$

But there are cases where D is a continuous lattice and this function is still not continuous, e.g., the domain of closed real intervals discussed below.

Construction of Domains

The interpretation described in the previous section is sufficiently flexible that it can be used to "construct" any domain. For suppose D_o is a complete lattice with a countable subbasis E_o. Let $\mathcal{M} = E_o$ and $\mathcal{U} = D_o$, and let $m \,\sigma\, u$ be $m \sqsubseteq u$. Then

$$\alpha(M) = \{\, u \mid u \in D_o \text{ and } \bigsqcup_{D_o} M \sqsubseteq_{D_o} u \,\} .$$

But since E_o is a subbasis of D_o,

$$\{\, \bigsqcup_{D_o} M \mid M \subseteq E_o \} = D_o \quad ,$$

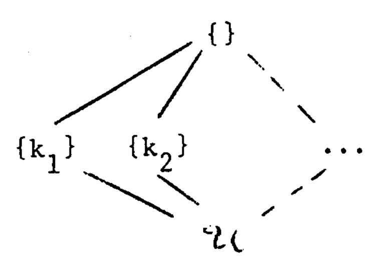
so that the image of α is the set

$$D = \{\{\, u \mid u \in D_o \text{ and } x \sqsubseteq_{D_o} u \,\} \mid x \in D_o \,\} .$$

which, when partially ordered by the inverse of set inclusion, is a lattice isomorphic to D_o.

In addition, the logical interpretation provides very natural constructions for a variety of simple domains. For example:

(1) Let $\mathcal{M} = \mathcal{U}$ be a countable set $\{k_1, k_2, \ldots\}$ and let σ be the relation of equality. If $\mathcal{M} = \mathcal{U}$ contains zero or one members, then D is a domain with the single element $\mathcal{U}$. Otherwise D is a primitive domain:



(2) Let $\mathcal{M} = \mathcal{U}$ be a countable, totally ordered set $\{k_1 < k_2 < \ldots\}$ and let $m \,\sigma\, u$ be $m < u$. Then D is a vertical domain:

$$\begin{array}{c} \{\ \} \\ \vdots \\ \{k \mid k > k_2\} \\ | \\ \{k \mid k > k_1\} \\ | \\ \mathcal{U} \end{array}$$

(3) Let $\mathcal{U} = 2^{\mathcal{M}}$, and let $m \,\sigma\, u$ be $m \in u$. Then

$$D = \{\{u \mid u \subseteq \mathcal{M} \text{ and } M \subseteq u\} \mid M \subseteq \mathcal{M}\}$$

with $\sqsubseteq\ =\ \supseteq$, which is isomorphic to the powerset domain $2^{\mathcal{M}}$.

(4) Let $\mathcal{U}$ be the set of real numbers, let $\mathcal{M}$ be the set of all closed non-empty real intervals which have finite rational endpoints, and let $m \,\sigma\, u$ be $u \in m$. Then D consists of all closed non-empty real intervals with finite endpoints, plus the empty interval and the entire real axis, with $\sqsubseteq\ =\ \supseteq$.

(5) Let $\mathcal{M} = A \times B$ and $\mathcal{U} = A \Rightarrow B$, where A and B are countable sets. Let $m \,\sigma\, u$ be $u([m]_1) = [m]_2$. Then D consists of sets of total functions from A to B which extend a particular partial function, plus the empty set. It is isomorphic to a domain which consists of the partial functions from A to B (with $\sqsubseteq$ the inverse of functional extension), plus a single overdetermined element which is approximated by every element.

(6) Suppose the domain D_1 is constructed from $\mathcal{M}_1$, $\mathcal{U}_1$, and σ_1, while D_2 is constructed from $\mathcal{M}_2$, $\mathcal{U}_2$, and σ_2. Let $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$ and $\mathcal{U} = \mathcal{U}_1 + \mathcal{U}_2$, where + denotes disjoint union, and let

$$m \; \sigma \; u = \begin{cases} m \; \sigma_1 \; u & \text{when } m \in \mathcal{M}_1 \text{ and } u \in \mathcal{U}_1 \\ m \; \sigma_2 \; u & \text{when } m \in \mathcal{M}_2 \text{ and } u \in \mathcal{U}_2 \\ \text{true} & \text{when } m \in \mathcal{M}_1 \text{ and } u \in \mathcal{U}_2 \\ \text{true} & \text{when } m \in \mathcal{M}_2 \text{ and } u \in \mathcal{U}_1 \end{cases}$$

Then

$$D = \{ \, d_1 + d_2 \mid d_1 \in D_1 \text{ and } d_2 \in D_2 \, \} ,$$

which is isomorphic to the lattice product $D_1 \times D_2$.

Further Possibilities

The obvious limitation of the interpretations we have presented is that a message sequence is always treated as an enumeration, so that its meaning must be independent of the order of the messages. The possibility of relaxing this restriction seems a fruitful line for further research. It might also be fruitful to seek an interpretation which makes lattice continuity more inevitable.

ACKNOWLEDGEMENTS

The author would like to thank R. Burstall and D. Scott for stimulating discussions and helpful questions.

APPENDIX

Proposition 1 If X is a directed subset of D, then $x \prec \bigsqcup X$ if and only if $x \prec y$ for some $y \in X$.

Proof: Left to the reader.

Proposition 2 $\{x \mid x \in D \text{ and } x \prec y\}$ is a directed set.

Proof: Left to the reader.

Proposition 3 If $f \in D \Rightarrow D'$ and D is a continuous lattice, then f is a continuous function if and only if, for all $y \in D$,

$$f(y) = \bigsqcup\{ f(x) \mid x \in D \text{ and } x \prec y \} .$$

Proof: Suppose f is continuous. By the continuity of D and Proposition 2 we have

$$f(y) = f(\bigsqcup\{ x \mid x \in D \text{ and } x \prec y \}) = \bigsqcup\{ f(x) \mid x \in D \text{ and } x \prec y \}.$$

On the other hand, suppose that the equation in the proposition is true, and let $X \subseteq D$ be directed. Then by Proposition 1, we have

$$\begin{aligned} f(\bigsqcup X) &= \bigsqcup\{ f(x) \mid x \in D \text{ and } x \prec \bigsqcup X \} \\ &= \bigsqcup\{\bigsqcup\{ f(x) \mid x \in D \text{ and } x \prec y \} \mid y \in X \} \\ &= \bigsqcup\{ f(y) \mid y \in X \} . \end{aligned}$$

Proposition 4 If $f \in D \rightarrow D'$ is additive, and $R' \subseteq D'$ is the range of f, then R' is a complete lattice with the same partial ordering as D'. Moreover, if E is a subbasis (basis) of D then the image of E under f is a subbasis (basis) of R'.

Proof: To show that R' is a complete lattice, it is sufficient to show that $\bigsqcup_{D'} X' \in R'$ whenever $X' \subseteq R'$. But if X' is a subset of the range of f, then there is a $X \subseteq D$ such that $X' = \{ f(x) \mid x \in X \}$, so that the additivity of f gives $\bigsqcup X' = \bigsqcup \{ f(x) \mid x \in X \} = f(\bigsqcup X) \in R'$. (Note that, although $\bigsqcup_{R'} X' = \bigsqcup_{D'} X'$, it is possible that $\bigsqcap_{R'} X' \neq \bigsqcap_{D'} X'$, so that R' may not be a sublattice of D'.)

Let E' be the image of E under f. If E is a subbasis of D, then for any $x' \in R'$ there is an $x \in D$ such that

$$\begin{aligned} x' = f(x) &= f(\bigsqcup \{ e \mid e \in E \text{ and } e \sqsubseteq x \}) \\ &= \bigsqcup \{ f(e) \mid e \in E \text{ and } e \sqsubseteq x \} \\ &\sqsubseteq \bigsqcup \{ f(e) \mid e \in E \text{ and } f(e) \sqsubseteq f(x) \} \\ &= \bigsqcup \{ e' \mid e' \in E' \text{ and } e' \sqsubseteq x' \} . \end{aligned}$$

Since the opposite ordering is obvious, E' is a subbasis of R'.

If E is also a basis of D and X' is a finite subset of E', then X' is the image of some finite $X \subseteq E$. Then the additivity of f gives

$$\bigsqcup X' = \bigsqcup \{ f(x) \mid x \in X \} = f(\bigsqcup X) \in E',$$

so that E' is a basis of R'.

<u>Proposition 5</u> If $f \in D \to D'$ is an additive surjection, $g \in D' \Rightarrow D''$, and the composition $g \cdot f$ is a continuous function, then g is a continuous function.

Proof: Let X' be a directed subset of D', and let $X \subseteq D$ be the preimage of X' under f. Any finite $Y \subseteq X$ must have a finite image $Y' \subseteq X'$ which is bounded by some $z' \in X'$ which is the image of some $z \in X$. But since f is additive, $f(\bigsqcup(Y \cup \{z\}) = \bigsqcup(Y' \cup \{z'\}) = z'$, so that $\bigsqcup(Y \cup \{z\})$ is a member of X which bounds Y. Thus X is directed.

Then, by the continuity of f and g·f,

$$g(\bigsqcup X') = g(\bigsqcup\{ f(x) \mid x \in X \}) = g(f(\bigsqcup X))$$

$$= \bigsqcup\{ g(f(x)) \mid x \in X \} = \bigsqcup\{ g(x') \mid x' \in X' \},$$

so that g is continuous.

<u>Proposition 6</u> If $f \in D \to D'$ and $g \in D' \to D$ satisfy $g(f(x)) \sqsubseteq x$ for all $x \in D$, then for all $x \in D$ and $x' \in D'$, $x' \prec' f(x)$ implies $g(x') \prec x$.

Proof: If $x' \prec' f(x)$, then there is an open set U' such that $f(x) \in U'$, and $x' \sqsubseteq y'$ for all $y' \in U'$. Since f is continuous, the pre-image U of U' under f must also be open. But U contains x, and for all $y \in U$, $f(y) \in U'$ implies $x' \sqsubseteq f(y)$, which implies $g(x') \sqsubseteq g(f(y)) \sqsubseteq y$. Thus $g(x') \prec x$.

<u>Proposition 7</u> If $f \in D \to D'$ and $g \in D' \to D$ satisfy $g(f(x)) = x$ for all $x \in D$, and D' is a continuous lattice, then D is a continuous lattice.

Proof: For all $y \in D$, using Propositions 2 and 6, we have

$$y = g(f(y)) = g(\bigsqcup\{ x' \mid x' \in D' \text{ and } x' \prec f(y)\})$$

$$= \bigsqcup\{g(x') \mid x' \in D' \text{ and } x' \prec f(y)\}$$

$$\sqsubseteq \bigsqcup\{g(x') \mid x' \in D' \text{ and } g(x') \prec y\}$$

$$\sqsubseteq \bigsqcup\{x \mid x \in D \text{ and } x \prec y \}$$

The opposite ordering is obvious.

<u>Proposition 8</u> If $g \in D' \to D$ is an additive surjection and D is a continuous lattice, then the function $f \in D \Rightarrow D'$ such that

$$f(x) = \bigsqcup\{\ x' \mid x' \in D' \text{ and } g(x') \prec x\ \}$$

is a continuous function such that $g(f(x)) = x$ for all $x \in D$.

Proof: If $X \subseteq D$ is directed, then by Proposition 1,

$$\begin{aligned} f(\bigsqcup X) &= \bigsqcup\{\ x' \mid x' \in D' \text{ and } g(x') \prec \bigsqcup X\ \} \\ &= \bigsqcup\{\bigsqcup\{\ x' \mid x' \in D' \text{ and } g(x') \prec x\ \} \mid x \in X\ \} \\ &= \bigsqcup\{\ f(x) \mid x \in X\ \} \end{aligned}$$

so that f is continuous.

If $x \in D$, then by the continuity of D, and the surjectiveness and additivity of g,

$$\begin{aligned} x &= \bigsqcup\{\ z \mid z \in D \text{ and } z \prec x\ \} \\ &= \bigsqcup\{\ g(x') \mid x' \in D' \text{ and } g(x') \prec x\ \} \\ &= g(\bigsqcup\{\ x' \mid x' \in D' \text{ and } g(x') \prec x\ \}) \\ &= g(f(x)). \end{aligned}$$

BIBLIOGRAPHY

The following is a list of references on the theory and application of the lattice-theoretic method which we have tried to make as complete as possible. A concise presentation of the basic theory is given in (4); more elementary expositions are given in (7) and (12).

1. Scott, D. "Outline of a Mathematical Theory of Computation," Proc. Fourth Annual Princeton Conf. on Information Sciences and Systems (1970), pp. 169-176. Also, Tech. Monograph PRG-2, Programming Research Group, Oxford University Computing Laboratory, November 1970.

2. _____. "The Lattice of Flow Diagrams," Symposium on Semantics of Algorithmic Languages, Ed. E. Engeler, Springer Lecture Note Series No. 188, Springer-Verlag, Heidelberg (1971), pp. 311-366. Also, Tech. Monograph PRG-3, Programming Research Group, Oxford University Computing Laboratory, November 1970.

3. _____. "Lattice Theory, Data Types, and Semantics," New York University Symposia in Areas of Current Interest in Computer Science, Ed. R. Randall (1971).

4. _____. "Continuous Lattices," Proc. 1971 Dalhousie Conf., Springer Lecture Note Series, Springer-Verlag, Heidelberg. Also, Tech. Monograph PRG-7, Programming Research Group, Oxford University Computing Laboratory, August 1971.

5. _____. "Lattice-Theoretic Models for Various Type-Free Calculi," Proc. Fourth International Congress for Logic, Methodology, and the Philosophy of Science, Bucharest (1972).

6. _____. "Mathematical Concepts in Programming Language Semantics," AFIPS Conference Proc., vol 40, AFIPS Press, Montvale, New Jersey (1972), pp. 225-234.

7. _____. "Data Types as Lattices", Notes, Amsterdam, June 1972.

8. _____, and Strachey, C. "Towards a Mathematical Semantics for Computer Languages," Proc. Symposium on Computers and Automata, Microwave Research Institute Symposia Series, Vol. 21, Polytechnic Institute of Brooklyn (1972). Also, Tech. Monograph PRG-6, Programming Research Group, Oxford University Computing Laboratory, August 1971.

9. Wadsworth, C. P., "Semantics and Pragmatics of the Lambda-Calculus", Ph.D. Thesis, Oxford University, September, 1971.

10. Milner, R. "Implementation and Applications of Scott's Logic for Computable Functions," Proc. ACM Conf. on Proving Assertions about Programs, SIGPLAN Notices, Vol. 7, No. 1, (or SIGACT News, No. 14), January 1972, pp. 1-6.

11. Strachey, C. "Varieties of Programming Language," Proc. ACM Conf. Venice (1972) (to appear).

12. Reynolds, J. C., "Notes on a Lattice-Theoretic Approach to the Theory of Computation", Systems and Information Science, Syracuse University, October 1972.

13. Kahn, G. "A Preliminary Theory for Parallel Programs", unpublished.