15-852: COMPUTATIONAL GEOMETRY **Lecturer:** Gary Miller
Topic: Geometric Transforms **Date:** October 25, 2006
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1 Geometric Transforms

The objective of the lecture is to present the intimate relations and connections between the **Convex Hull** problem and several other problems: **Linear Programming**, **Delaunay Triangulation**, **Voronoi Diagrams**, **Stereographic Maps**.

Throughout the lecture, we denote by d the dimension of the space we are working in. So, we can consider we are in the Euclidean space $\mathbb{R}^d$. Although we're mainly interested in the case $d = 2$, unless we do not explicitly say so, the claims we make will hold true for any positive integer value d. We shall regard any point $p \in \mathbb{R}^d$ as the vector

$$p = \begin{bmatrix} p_1 & p_2 & \cdots & p_d \end{bmatrix}^T$$

Then, the square of p's Euclidean norm (or "2-norm") is

$$||p||_2^2 = \sum_{i=1}^{d} p_i^2 = p^T p = \begin{bmatrix} p_1 & p_2 & \cdots & p_d \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_d \end{bmatrix}$$

Definition 1.1 *A $d-1$-**dimensional sphere** in $\mathbb{R}^d$ is defined as the set of points x lying at distance $r \geq 0$ from some center c, and can be specified algebraically as $\{x \in \mathbb{R}^d | (x-c)^T(x-c) = r^2\}$. By unit sphere, we mean the sphere centered at the origin with radius 1: $\{x \in \mathbb{R}^d | x^T x = 1\}$.*

Note 1.2 *In the degenerate case when the radius is infinite, a sphere becomes a hyperplane and it can be specified algebraically as $\{x \in \mathbb{R}^d | a^T x = b\}$, where $a \in \mathbb{R}^d$, $b \in \mathbb{R}$.*

Claim 1.3 *Let $S = \{x \in \mathbb{R}^d | \alpha x^T x - 2b^T x + \beta = 0\}$, where $\alpha^2 + b^T b > 0$ (that is, α and b can't be zero in the same time). Then S is a sphere iff $\alpha\beta \leq b^T b$.*

Proof: In case $\alpha = 0$, we're done. Indeed, then $b \neq 0$ and $S = \{x \in \mathbb{R}^d | -2b^T x + \beta = 0\}$ is a hyperplane (and so - a sphere).

In case $\alpha \neq 0$, we can assume w.l.o.g. $\alpha = 1$. It remains to show that S sphere $\Leftrightarrow \beta \leq b^T b$. Clearly, we have:

$S = \{x \in \mathbb{R}^d | x^T x - 2b^T x + \beta = 0\} = \{x \in \mathbb{R}^d | x^T x - 2b^T x + b^T b = b^T b - \beta\}$

If $\beta \leq b^T b$, it follows S is a sphere with $c = b$ and $r = \sqrt{b^T b - \beta}$. On the other hand, if S is a sphere, then the square of the radius has to be non-negative, i.e. $\beta \leq b^T b$. ■

Exercise 1.4 *Let* $S = \{x \in \mathbb{R}^d | \alpha x^T x - 2b^T x + \beta = 0\}$. *Prove that:*

1. *if* $\alpha^2 + b^T b = 0$, *then* $S = \mathbb{R}^d$.
2. *if* $\beta^2 + b^T b = 0$, *then either* $S = \{0\}$ *or* $S = \mathbb{R}^d$.

2 Reflection about the unit sphere

Definition 2.1 *The reflection of* $p \in \mathbb{R}^d \backslash \{0\}$ *about the unit sphere is, by definition the point*

$$Reflect(p) = \frac{p}{p^T p}$$

By convention, $Reflect(0)$ *is the point at the infinite.*

Note 2.2

1. *If* p *lies on the unit sphere, then so does* $Reflect(p)$ *(moreover,* $Reflect(p) = p$*).*
2. *If* p *belongs to the interior of the unit sphere, then* $Reflect(p)$ *lies outside, and viceversa.*

Proposition 2.3 *For any* $p \in \mathbb{R}^d\ \{0\}$, $Reflect(Reflect(p)) = p$ *and* $p^T Reflect(p) = 1$.

Proof:

$$Reflect(Reflect(p)) = \frac{Reflect(p)}{Reflect(p)^T Reflect(p)} = \frac{\frac{p}{p^T p}}{\frac{p^T}{p^T p} \frac{p}{p^T p}} = \frac{\frac{p}{p^T p}}{\frac{p^T p}{(p^T p)^2}} = p$$

$$p^T Reflect(p) = p^T \frac{p}{p^T p} = \frac{p^T p}{p^T p} = 1$$

■

Theorem 2.4 *The reflection about the unit sphere maps spheres to spheres.*

Proof: Let $S = \{x \in R^d | \alpha x^T x - 2b^T x + \beta = 0\}$ be a sphere. Assume now that $\alpha^2 + b^T b > 0$ and $\beta^2 + b^T b > 0$ (it's easy to treat the opposite situation).

Let $S' = \{Reflect(x) | \alpha x^T x - 2b^T x + \beta = 0\}$. But then (assuming $x \neq 0$):

$$\alpha Reflect(x)^T Reflect(x) - 2b^T Reflect(x) + \beta = 0 \Leftrightarrow \alpha \frac{x^T x}{(x^T x)^2} - 2b^T \frac{x}{x^T x} + \beta = 0 \Leftrightarrow$$

$$\frac{\alpha}{x^T x} - 2\frac{b^T x}{x^T x} + \beta = 0 \Leftrightarrow \alpha - 2b^T x + \beta x^T x = 0.$$

We know that this represents a sphere iff $\alpha\beta \leq b^T b$. But the last inequality happens, because S is a sphere.

■

3 The Dual Transform

Definition 3.1 *For any $p \in \mathbb{R}^d$, we define the halfspace associated to p, as:*

$$HS_p = \{x \in \mathbb{R}^d | p^T x \leq 1\}$$

Using this definition, we shall try to prove that the finding the solution to the Convex Hull of a set of points, $P = \{p^{(1)}, \cdots, p^{(m)}\}$ is equivalent to finding the boundary of the polytope $\bigcap_{p \in P} HS_p$. (Actually, we claim these two objects are isomorphic.)

Note 3.2 *1. $Reflect(p) \in HS_p$ (cf. Proposition 2.3).*

2. $\forall p \in \mathbb{R}^d$, HS_p contains the origin of the space.

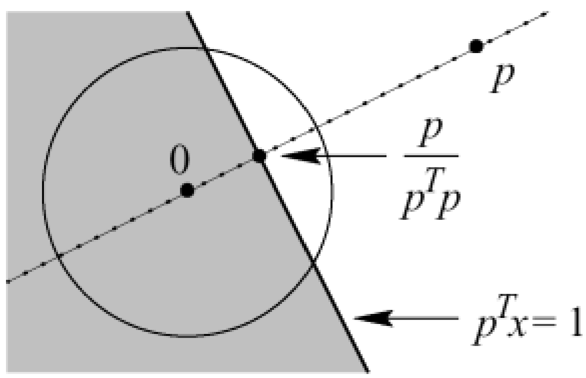

Figure 1. The Dual Transform

Claim 3.3 *As p gets further and further out (i.e. as its norm increases), HS_p gets further and further towards the origin, and thus constitutes a stronger constraint.*

Proof: Exercise. ■

Definition 3.4 *If $p \in P$, we say that HS_p is a critical halfspace if $\exists r \in \partial(HS_p)$ (i.e. $p^T r = 1$) s.t. $\forall q \in P, q \neq p$, r is interior to HS_q.*

Definition 3.5 *A point p is a critical point if $p \in CH(P)$, and $\exists HS$ a halfspace with $p \in \partial(HS)$, s.t. $\forall q \in P$, $q \neq p$, q is interior to HS.*

Claim 3.6 *If P is a finite set of points whose convex closure contains zero then HS_p is critical iff p is a critical point.*

Proof: ($\rightarrow$) Assume that HS_p is critical then $\exists r$ s.t. $p^T r = 1$, $q \in P$, $q \neq p$, $q^T r < 1$. Since $p^T r = 1$ we sure have $p \in \partial(HS_r)$. For any point $q \in P$, $q \neq p$, q is interior to HS_r iff $q^T r < 1$ iff r is interior to HS_q.

($\leftarrow$) Assume that p is a critical point and HS is a supporting hyperplane. Since zero is in the convex hull of P we know that zero in contained to HS. Let r' be the closest point to zero on $\partial(HS)$ and $r = Reflect(r')$. It follows that $HS_r = HS$. We claim that HS_p is critical and r is the witness since $p^T r = 1$ and $q^T r < 1$ for $q \neq p$. ∎

4 The Stereographic Projection/Map

The stereographic projection is a function which maps points on the 2D sphere, of center $(0,0,0.5)^T$ and diameter 1 (i.e. radius 0.5) onto points in the 2-dimensional plane ($\mathbb{R}^3$) $\alpha_0 : z = 0$. Let's note that α_0 touches the sphere at its south pole (the origin), while the sphere's north pole is the point $(0,0,1)^T$. Each point p on the sphere except the north pole, is mapped to the plane as follows: Draw a ray starting at the north pole and passing through p; extend the ray until it intersects the plane; map p to this point on the plane. This is illustrated in Figure 2.

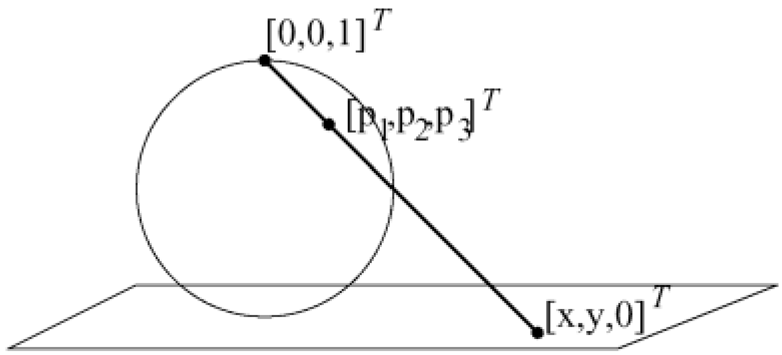

Figure 2. The Stereographic Map

Now, we'll give a more rigorous definition of the Stereographic Map (SM):

Definition 4.1 *Let* $q \in S_1 = \{(x,y,z)^T \in \mathbb{R}^3 | x^2+y^2+(z-0.5)^2 = (0.5)^2\}$. *Then* $SM(q) = (x,y)$, *where:*

$q \mapsto p = q - (0,0,1)^T \mapsto p' = Reflect(p) \mapsto (x,y,0)^T = p' + (0,0,1)^T \mapsto (x,y)^T$

Exercise 4.2 *Prove that the two definitions, the more intuitive one and the formal one, are in fact equivalent. (Hint: see Figure 3.)*

From the previous definition, it follows that SM is obtained by composing a translation with the reflection about the unit sphere, then again with a translation, and finally with a canonical inclusion.

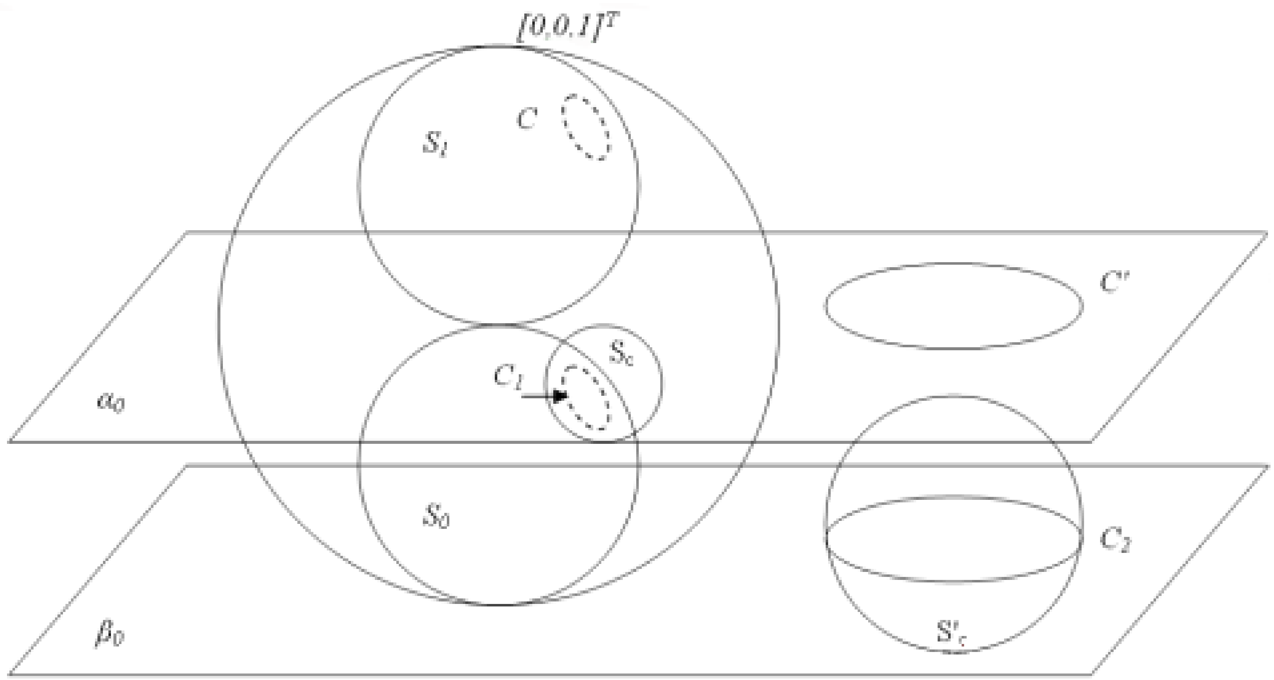

Figure 3. The Stereographic Map is conform

Theorem 4.3 *SM maps circles (2D spheres) on the sphere S_1, to circles on the plane α_0.*

Proof: (Sketch) Observe that $Reflect$ maps the plane $\beta_0 = \alpha_0 - (0, 0, 1)^T$, to the sphere $S_0 = S_1 - (0, 0, 1)^T$, and of course, $Reflect\,(S_0) = \beta 0$.

Let's consider now $C \subseteq \alpha_0$ an arbitrary circle on S_1. Let C_1 be the circle corresponding to C, on the sphere S_0. Let S_c be a sphere in $\mathbb{R}^3$ (different from S_0), that and includes $C1$. Then obviously, $C_1 = S_c \bigcap S_0$.

Use then Theorem 2.4 to show that $Reflect$ maps S_c onto another sphere S'_c. But since $Reflect\,(C_1) = Reflect\,(S_c \bigcap S_0) = Reflect\,(S_c) \bigcap Reflect\,(S_0) = S'_c \bigcap \beta_0$.

Since $C_1 \neq \emptyset$, it follows that $\emptyset \neq Reflect\,(C_1) = S'_c \bigcap \beta_0$. But the intersection of a sphere and a plane (when nonempty), is a circle. Call this C_2.

After translating this circle "up" one unit, we finally reach the conclusion that C is mapped by SM onto a circle on α_0. ∎

Exercise 4.4 *Show that SM^{-1}, the transform which is the inverse of SM, maps circles on α_0 to circles on S_1.*

5 The Parabolic Map

A transform which is related to the Stereographic Map is the Parabolic Map.

Definition 5.1 *The Parabolic map is a transform from* $\mathbb{R}^d$ *to* $\mathbb{R}^{d+1}$*, such that, for any* $p \in \mathbb{R}^d$*,* $Para\,(p) = \begin{bmatrix} p^T \\ p^T p \end{bmatrix}$.

This transform is not conform, but the image of a circle is still planar:

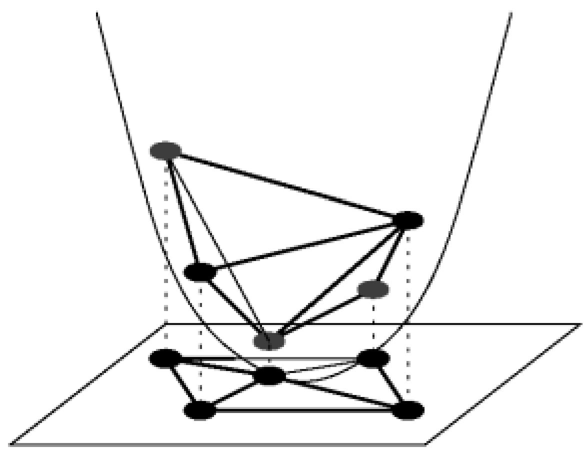

Figure 4. The Parabolic Map

Theorem 5.2 *If* S *a sphere in* $\mathbb{R}^d$ *then* $Para\,(S)$ *is coplanar.*

Proof: Let $S = \{x \in \mathbb{R}^d | \alpha x^T x - 2b^T x + \beta = 0\}$, with $\alpha\beta \leq b^T b$.

We know: $Para\,(S) : \{\begin{pmatrix} x \\ z \end{pmatrix} | x^T x = z\}$.

Let $x \in S$, $y = Para\,(x) = \begin{pmatrix} x \\ z \end{pmatrix}$. Then $\alpha z - 2b^T x + \beta = 0 \Rightarrow \begin{pmatrix} -2b \\ \alpha \end{pmatrix}^T y + \beta = 0$.

So, all y's belong to the same hyperplane.

■

6 Voronoi Diagrams and Delaunay Triangulations

Definition 6.1 *Let* $P \subseteq \mathbb{R}^d$*. Then, for* $p \in P$*, we define:*

$V_P(p) = \{x \in \mathbb{R}^d | \forall q \in P, q \neq p, dist(p,x) \leq dist(q,x)\}$.

In this case, p is called the centroid of $V_P(p)$.

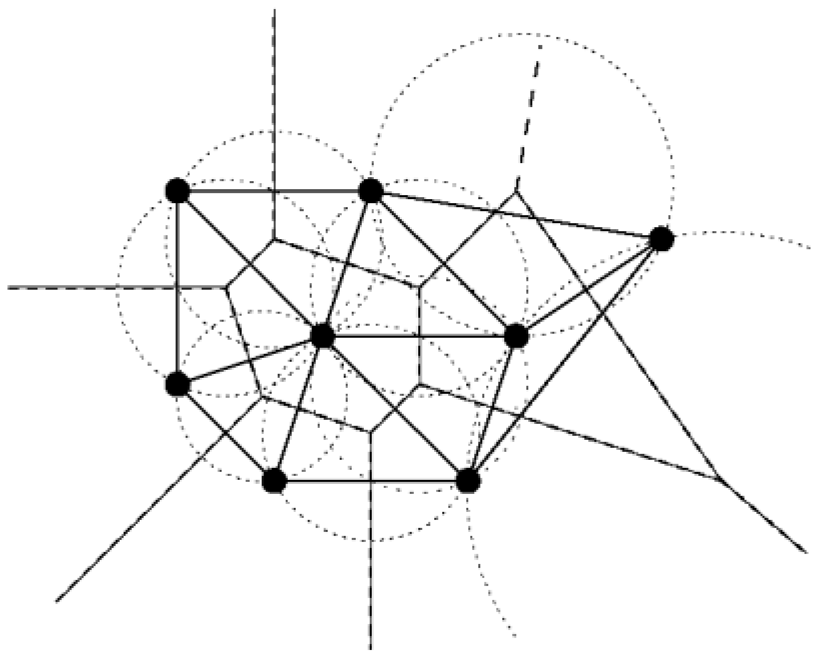

Figure 5. The Voronoi diagram and the Delaunay Triangulation are dual to each other

Claim 6.2 $V_P(p)$ *defined as above, is a convex polytope.*

Proof: For $q, p \in P$, $q \neq p$, let's define $HS_{p,q} = \{x | dist(p,x) \leq dist(q,x)\}$.

Then, obviously: $V_P(p) = \bigcap_{q \in P, q \neq p} HS_{p,g}$. Since $V_P(p)$ is the intersection of halfplanes, it is a convex polytope. ■

Definition 6.3 *The Voronoi diagram of a set P is the partition* $Vor(P) = (V_P(p))_{p \in P}$ *of* $\mathbb{R}^d$.

Note 6.4 *A point lying on the boundaries of several members of the Voronoi diagram of P, is equally far from the centroids of those members. In particular, in 2D, the intersection point of the boundaries of 3 faces is the circumcenter of the corresponding centroids.*

Definition 6.5 *Assume $P \subseteq \mathbb{R}^2$ a set of points in general position, i.e. no 4 points are co-circular. Then $Del(P)$ is a triangulation on P, such that for any triangle $\Delta(p,q,r) \in Del(P)$ $(p,q,r,\in P)$ the interior of the circumcircle of $\Delta(p,q,r)$ doesn't contain any point of P.*

We sometimes call this: "every triangle in the Delaunay triangulation has an empty circumcircle".

Claim 6.6 *$Vor(P)$ is the dual of $Del(P)$ (as planar graphs).*

Claim 6.7 *$Del(P) = CH(SM(P))$.*

Claim 6.8 *If we map $Del(P)$ on the sphere (using SM^{-1}), the "empty" circles on the plane are mapped onto "empty" circles on the sphere.*

Claim 6.9 $Vor(P) \equiv CH\left(Dual\left(SM^{-1}(P)\right)\right)$

From the last 4 claims, it follows that any algorithm that solves any of these problems (Convex Hull, Stereographic Map. Voronoi Diagram, Delaunay Triangulation), also solves the others.