Representing Topological Structures Using Cell-Chains

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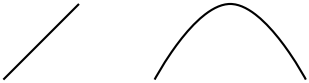
- 1-cells 

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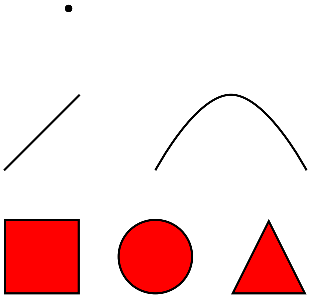

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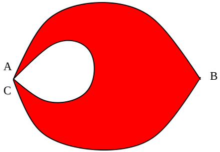
Pinched 2-cell:



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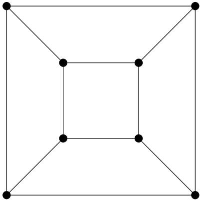

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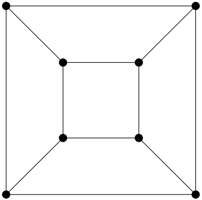

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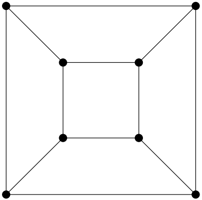

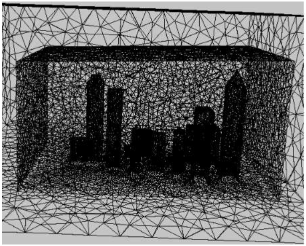

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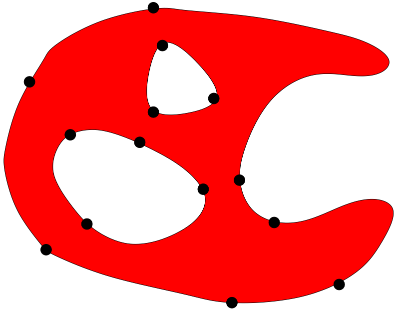

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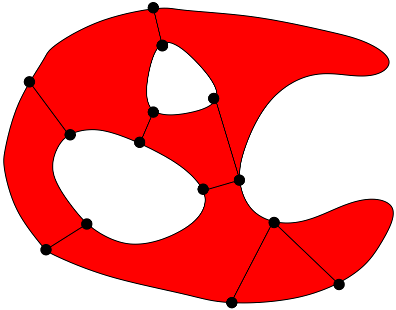

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- Space and Time efficient.

Data Structures Galore

Data Structure	Represents
Arcs *[Edmonds60]*	2D Surfaces
Crosses *[Tutte73]*	2D Unoriented Surfaces
Winged-Edge *[Bau75]*	3-d Polyhedra
Doubly-Linked Edge List *[MP78]*	Planar Subdivisions
Quad-Edge *[GS85]*	2D Surfaces
Facet-Edge *[DL87]*	Pseudo-Manifold Complexes
Radial-Edge *[Wei88]*	Non-Manifold B-Reps
Cell-Tuple-Complex *[Bri93]*	Regular Manifolds
nG-maps *[Lie94]*	Cellular Quasi-Manifolds
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Group Theoretic

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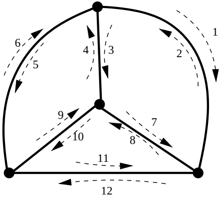
- 

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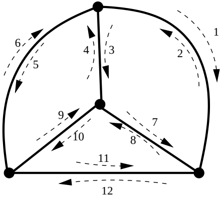
- 

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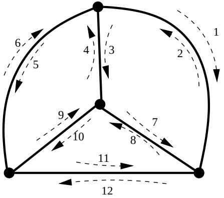
- 

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- $R^2 = id$ and fixed-point-free (fpf).

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- This is an Implicit representation!

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- Faces are represented implicitly in terms of operations on these objects.

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Holistic View

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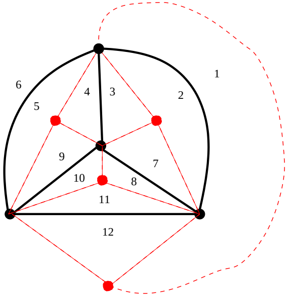

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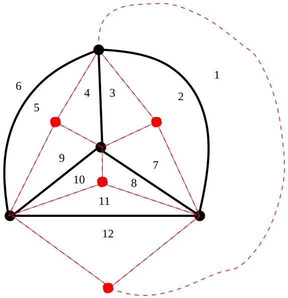

- The permutations now permute the new triangles.

Tutte's Representation of Surfaces

- An abstract set S of **crosses** and three permutations P, θ and ϕ of S satisfying:
- $\theta^2 = \phi^2 = I$ and $\theta\phi = \phi\theta$.
- X, θX, ϕX and $\theta\phi X$ are all distinct for all crosses X.
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- Modern View: Tutte had right group but wrong set of generators.

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- Cells should be "first class" objects.
- Represent more than two-dimensional decompositions.
- Rely on the same general principle; operators acting on very primitive objects.

The Cell-Complex Data Structure
[Bri93]

Brisson: Barycentric Subdivisions

Creating a **triangulation** of the cell complex .

- Add a new vertex interior to each edge and face. Connect two vertices if they are visible to each other.

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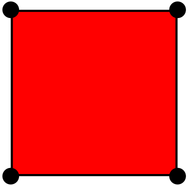
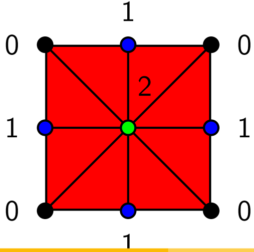
- 



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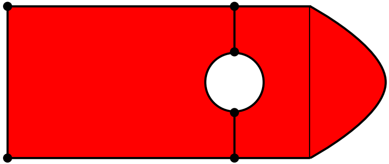
- 

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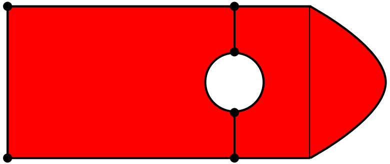
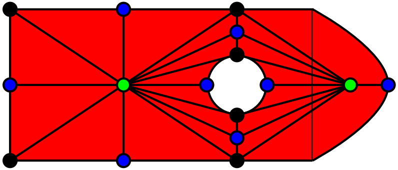
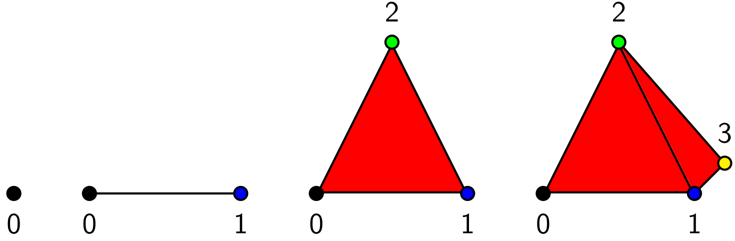
- 
- 

Numbered Simplices and Gluings

- A numbered d-simplex is a simplex whose vertices are uniquely labeled with numbers between 0 and d.



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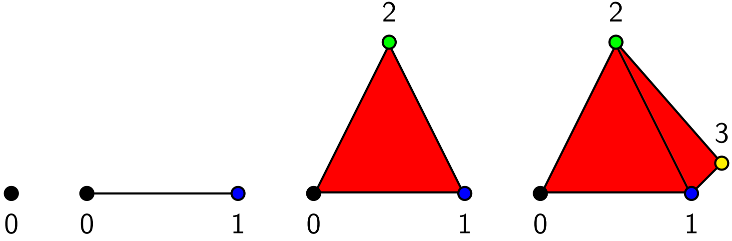

- Observe: The simplices in the Barycentric are numbered.

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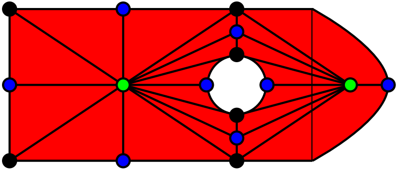
- 

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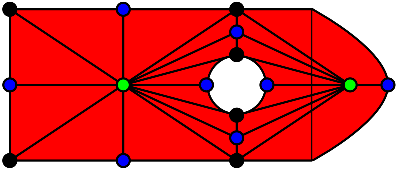
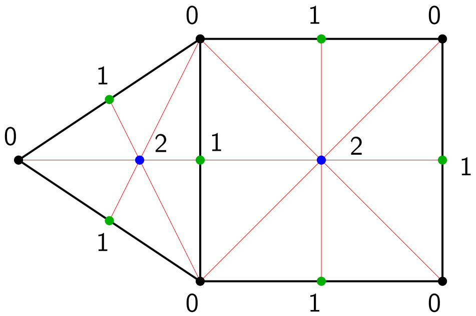
- 
- The set of gluings with the same label/numbers can be viewed as matching or permutation(involution).

Moving Around



- Simplex Reflection. Remove a vertex, take the simplex on the other side.

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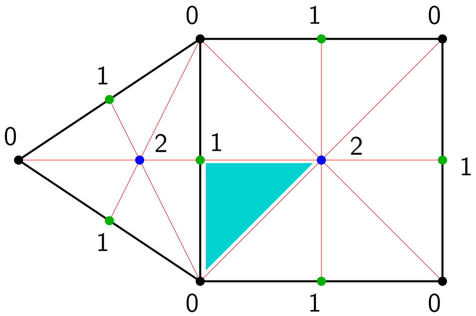

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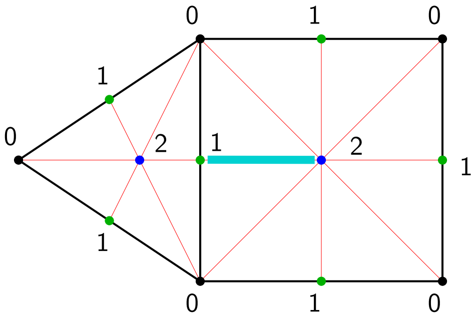

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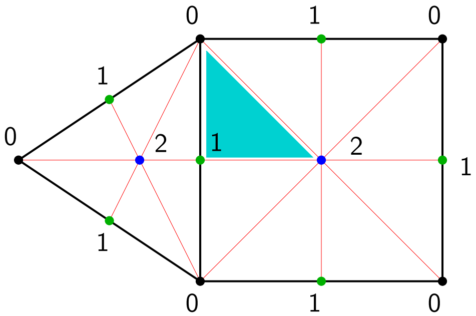

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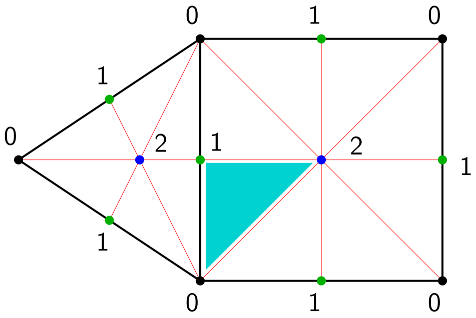

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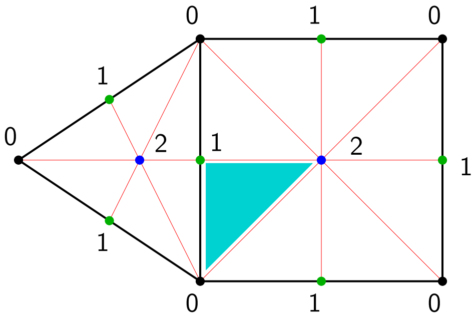

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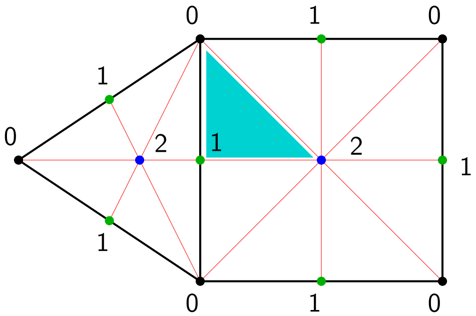

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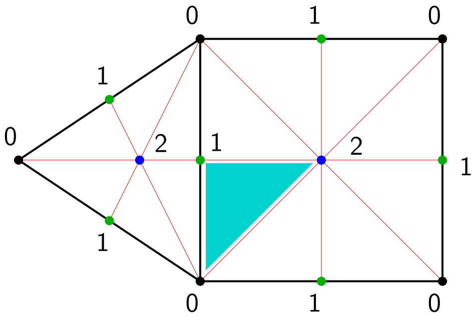

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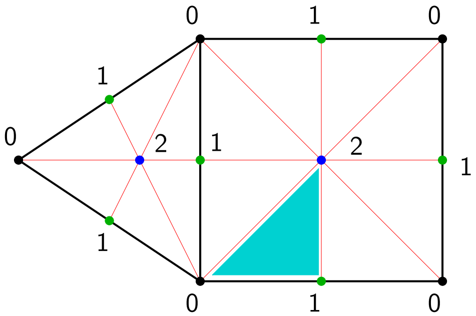

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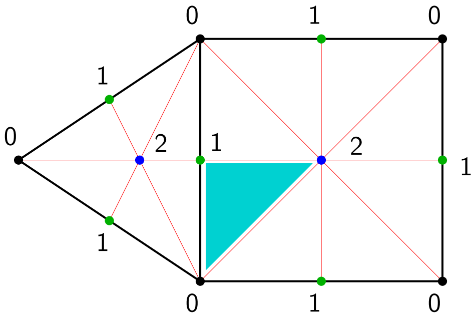

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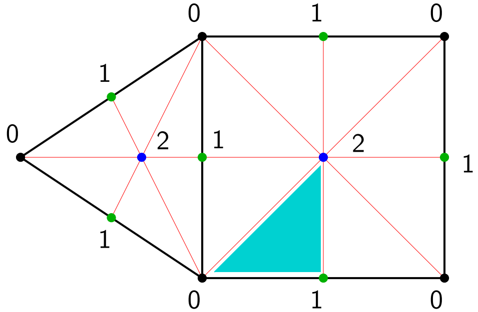

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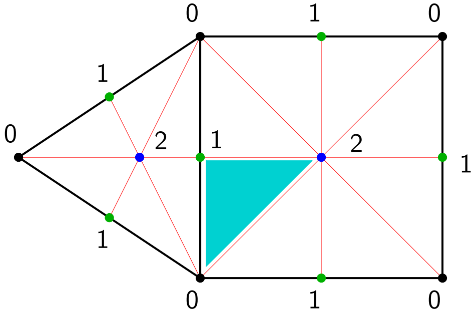

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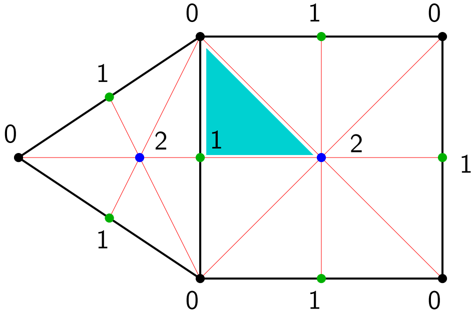

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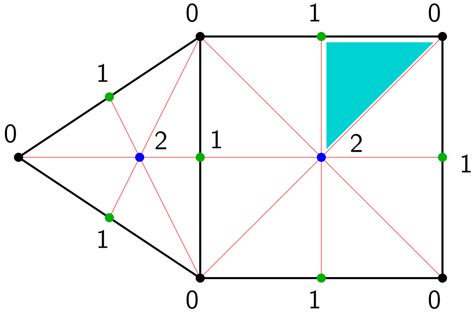

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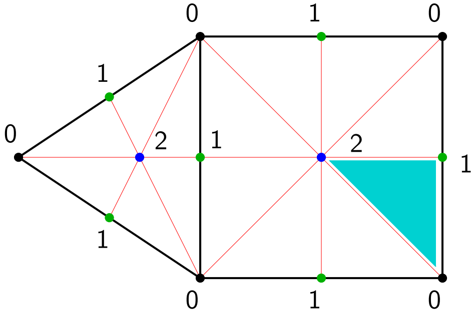

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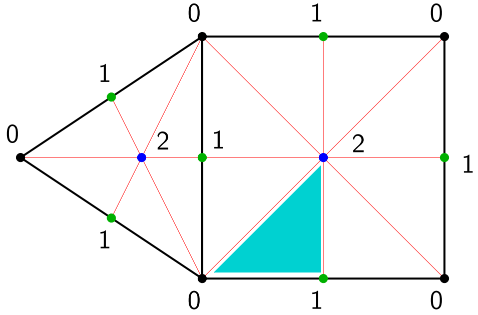

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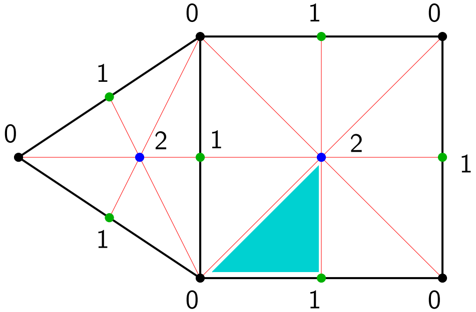

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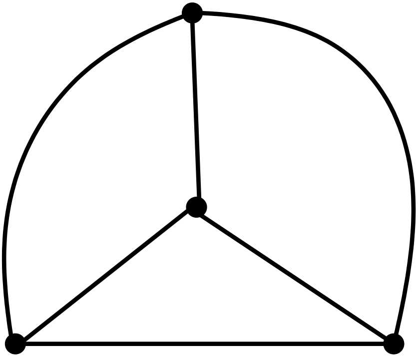
- Edges around a vertex? $\alpha_2\alpha_1$
- Faces around a face? $\alpha_2\alpha_1\alpha_0\alpha_2$
- Lots more in 3+ dimensions ...

A Modern Viewpoint of Tutte

His group is just a group of reflections the Barycentric simplices



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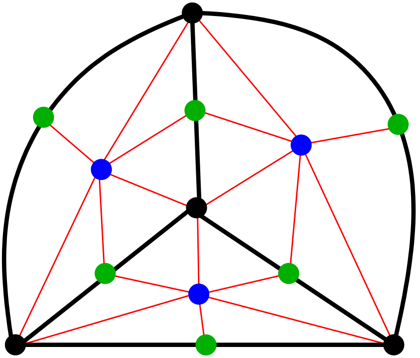
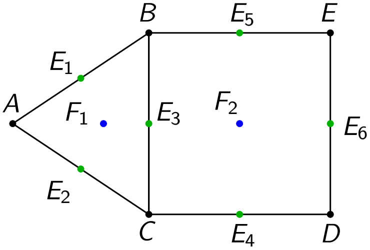
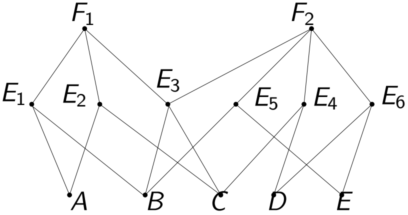


Incidence Graph and Cell-Tuples





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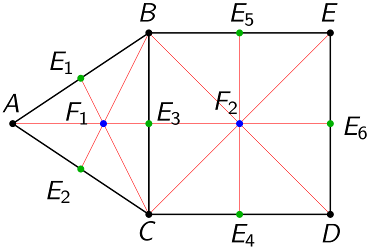

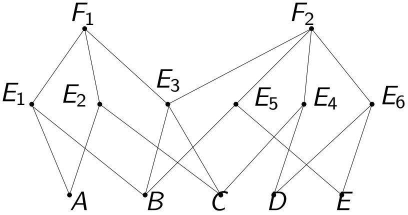

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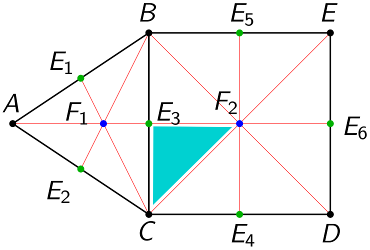

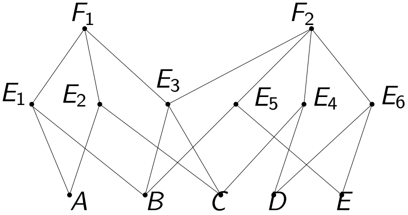

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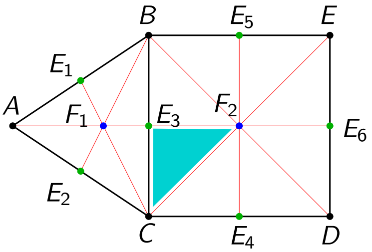

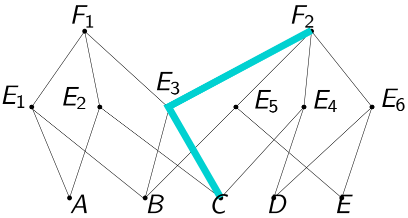

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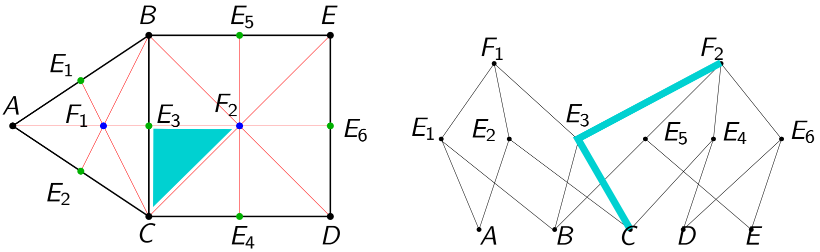

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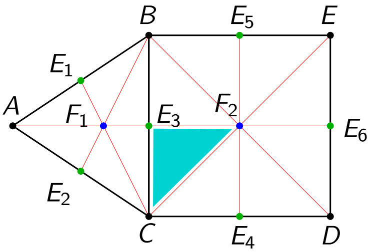

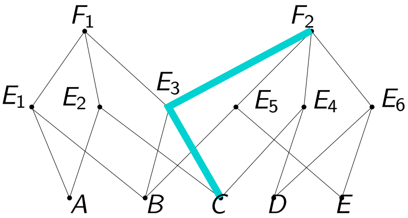

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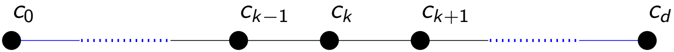
- **Thm:** Barycentric Simplices, Cell-Tuples, and Incidence Paths are all in a one-to-one correspondence.

Operating on Cell-Tuples with α_k



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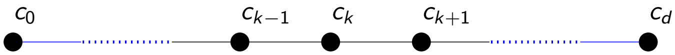

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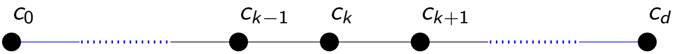

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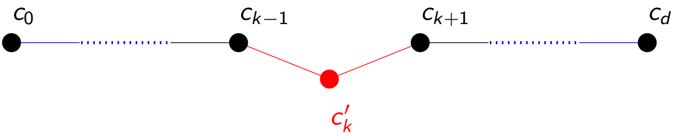

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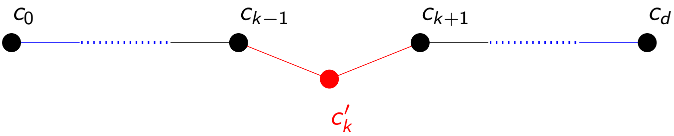

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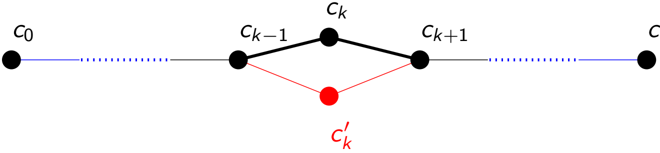
- $\alpha_k[T] = \langle c_0, \ldots, c_{k-1}, c'_k, c_{k+1}, \ldots, c_d \rangle$
- Furthermore, α_k is well-defined on the local portion of the Cell-Tuple:

$$\alpha_k \Big[\langle c_{k-1}, c_k, c_{k+1} \rangle \Big] = c'_k$$

Space savings using the switches α_k



One only stores all triples per switch.

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- For decidable semantics either admit non-regular manifolds or reject some.

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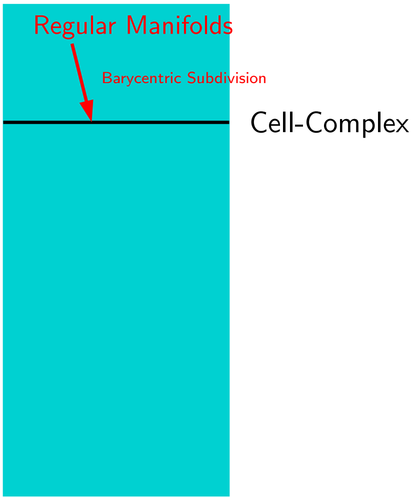
  - Characterize the class of cellular decompositions we can build.

Top-Down vs. Bottom-Up



Top-Down vs. Bottom-Up

Regular Manifolds

Barycentric Subdivision

Cell-Complex

Algebraic Construction

Recall: Permutation Group of a Set S

- Set of Permutation G of S.
- Closed: $\forall\ \alpha,\ \beta \in G,\ \alpha\beta \in G$
- Invertible: $\forall\ \alpha \in G,\ \exists\ \alpha^{-1} \in G$

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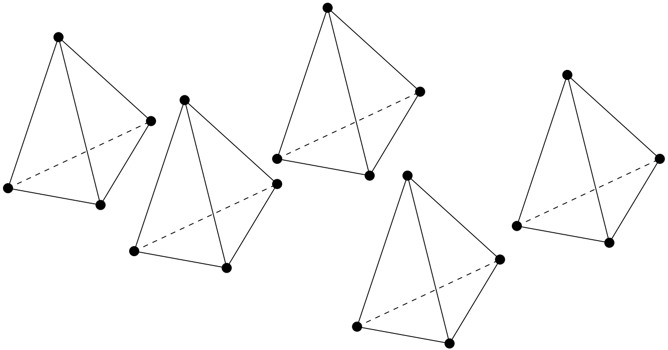
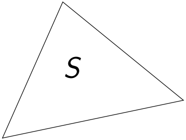
- S will be a set of numbered d-Simplices:



Basic Gluing Rules



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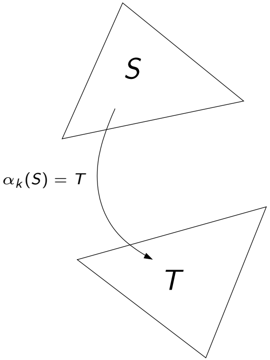

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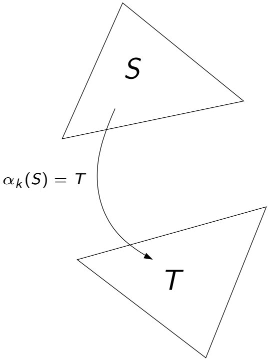

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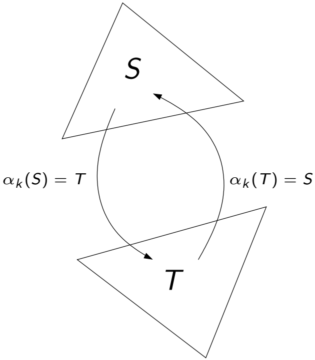

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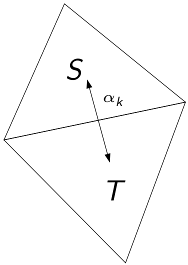

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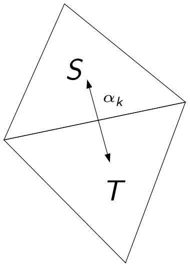

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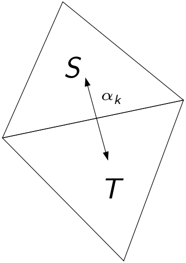

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- These properties define a **Map**.

A Commuting Map

- $\forall\ i, j \in 0 \ldots d,\ j \geq i + 2$

$$\alpha_i \alpha_j = \alpha_j \alpha_i$$

- Higher dimensional switches commute with lower dimensional switches.

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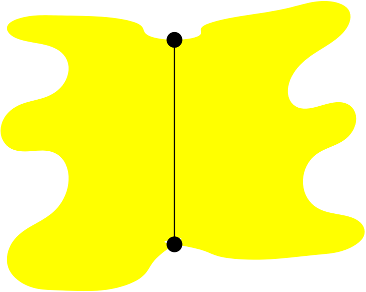

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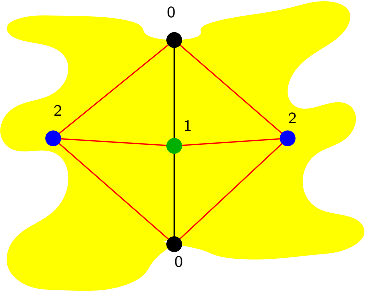

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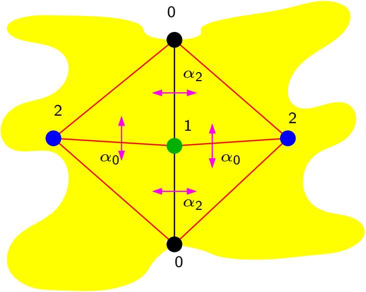

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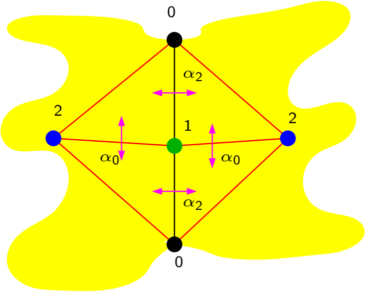

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- Also known as an *n*G-Maps *[Lie94]*

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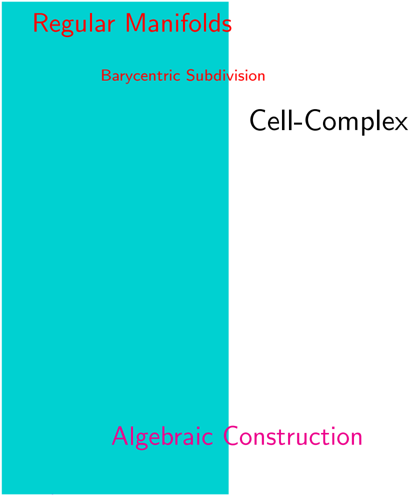

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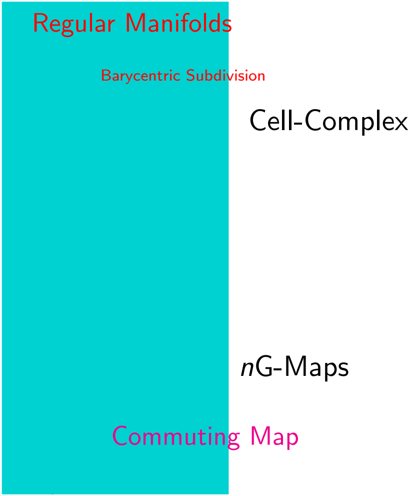

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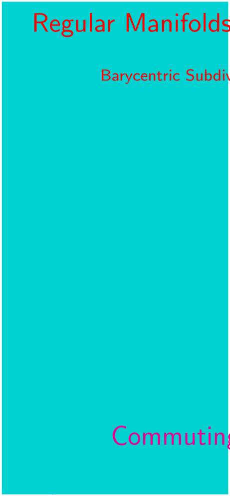

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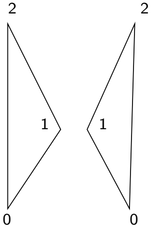

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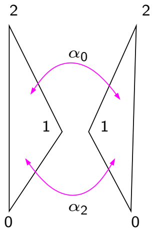

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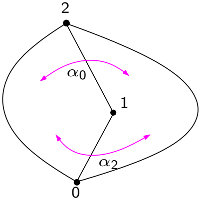

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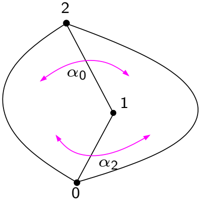

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- We need more restrictions on the gluing rules.

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$$\forall\ s \in S,\ k \in 1 \ldots (d-1)$$

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Orthogonality Axiom *[CMP]*

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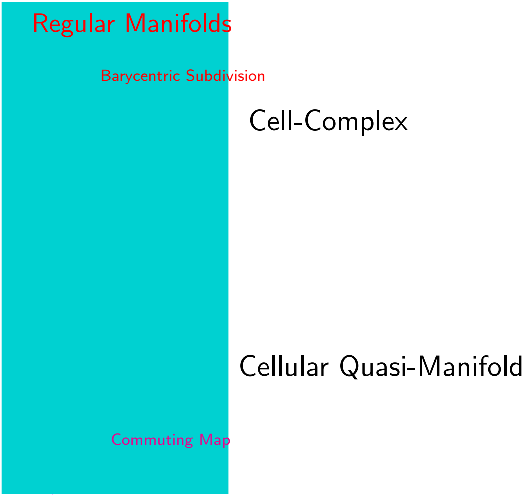
$$\forall\ s \in S,\ k \in 1 \ldots (d-1)$$

$$\forall\ \alpha \in \langle \alpha_0, \ldots, \alpha_{k-1} \rangle,\ \beta \in \langle \alpha_{k+1}, \ldots, \alpha_d \rangle,$$

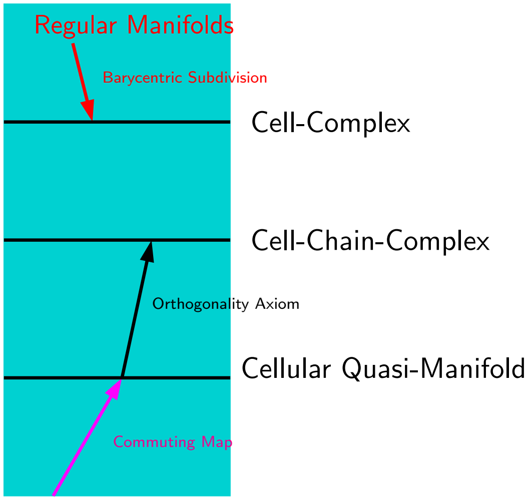
$$\alpha\beta(s) = s \longrightarrow \alpha(s) = \beta(s) = s$$

- Abstractly, higher and lower dimensional operators must not only commute, they must not interfere at all.
- Captures the idea that α_k can only affect the k^{th} entry.
- We call this a Cell-Chain-Complex.

Cell-Chain-Complex



Cell-Chain-Complex



Back to Data Structures

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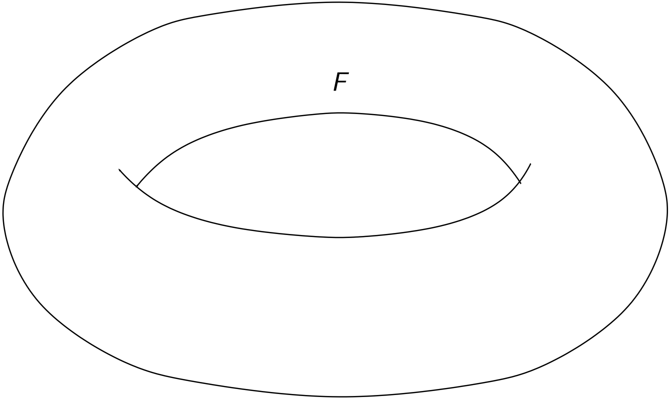

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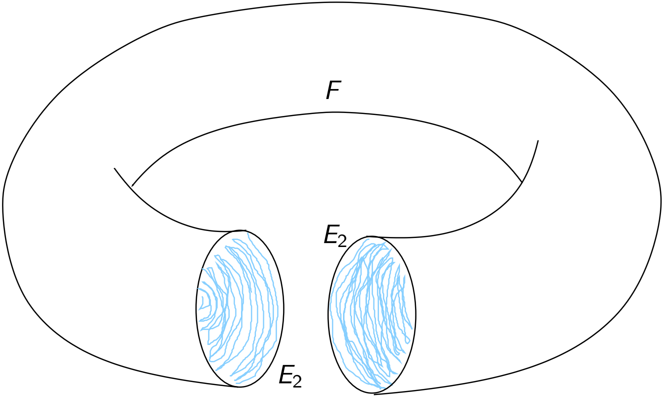

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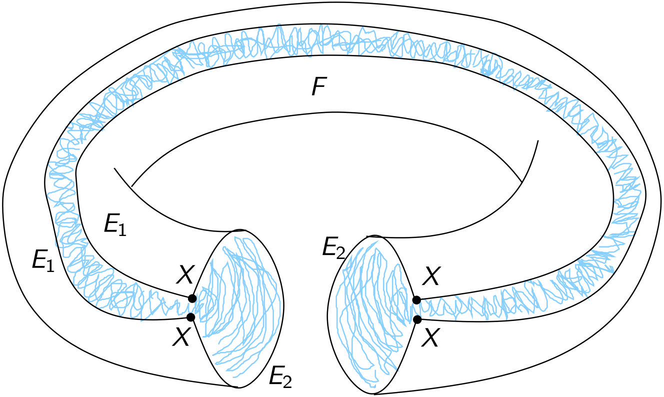

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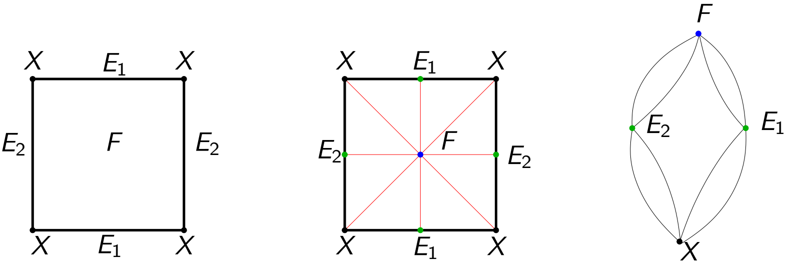
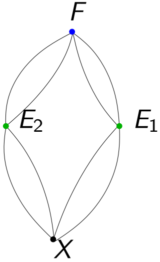


Edge Paths



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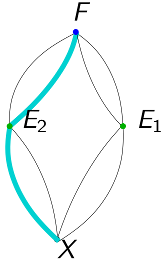

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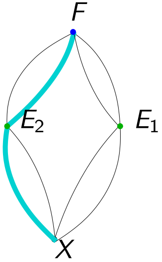

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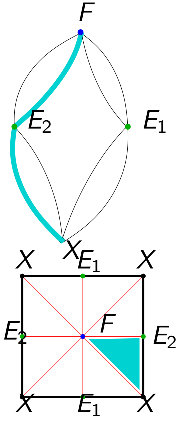

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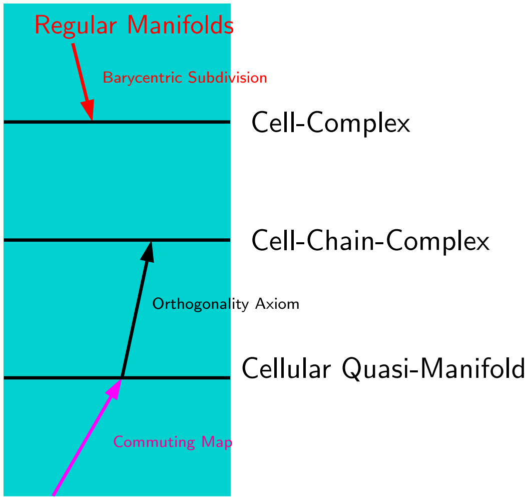
- **Thm(CMP):** In a Cell-Chain-Complex, Barycentric Simplices, Cell-Chains, and Incidence Edge-Paths are in one-to-one correspondence.

Bridging the Gap



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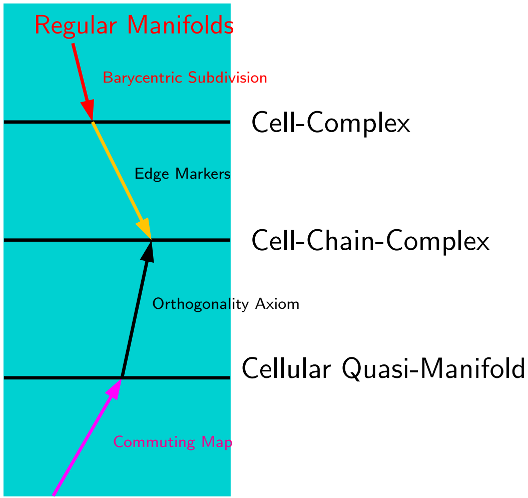

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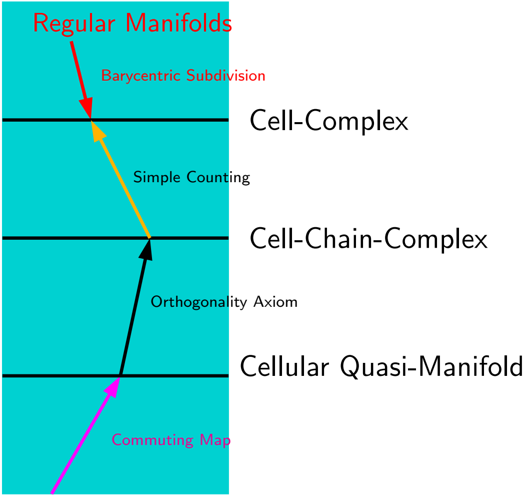

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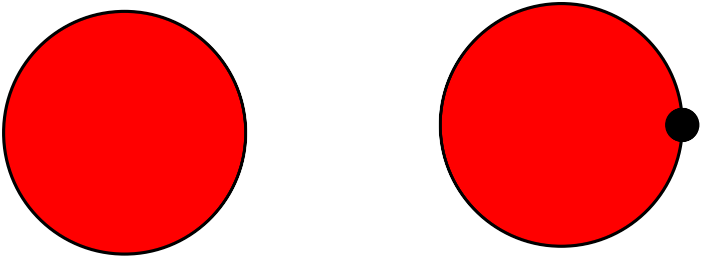
- The Barycentric Subdivision is a powerful tool for designing and understanding data structures.
- A Cell-Complex presents an elegant all-purpose data structure for cellular decompositions.
- A Cell-Chain-Complex successfully bridges the gap between rigorous algebraic structures and well-designed data structures.

Question

- Can we have a precise semantics for more degenerate cells?



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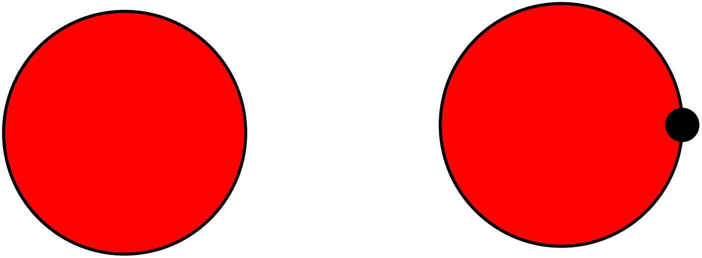
- In particular: A simple cycle has to include at least one point.



Question

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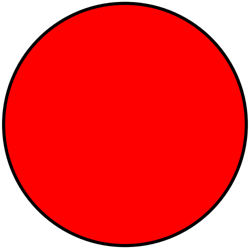
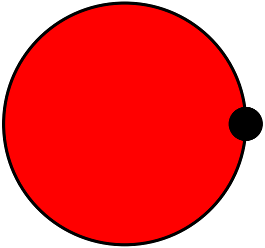
- In particular: A simple cycle has to include at least one point.
- Here we are modeling a 2D red blood cell.





Why a cycle should be point free.

The partial simulation of a 2D red blood red cell moving through a restriction. The boundary of the cell was given a simple closed loop containing one vertex. The vertex is on the right side of the image.