Well-Separated Pair Decomp (WSPD)

HP: Chap 3

We consider a decomp of points in $\mathbb{R}^d$ with many apps.

1) N-body simulations
2) Rep dist between all points via a sparse graph (Spanner)
3) 

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Def: A pair of sets of points $(Q, R)$ is $(\frac{1}{\epsilon})$-separated if

$$\max\{\text{dia}(Q), \text{dia}(R)\} \leq \epsilon \cdot d(Q, R)$$

\[ \text{Q} \quad \text{R} \]

\[ \text{d}(Q, R) \quad \text{d}(Q, R) \]
Since we will use a compressed QT to construct WSPD.

We take the following view:

1) We have a 2d-way tree $T$ with each $p \in P$ as a leaf.
2) Each pair is a link between two nodes of $T$.

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Note for each pair of points $p, q$

\[ \exists \text{ ancestor}(p) \uparrow \text{ ancestor}(q) \text{ that are linked (a pair).} \]

$u \in \mathcal{Z}$

$P_u = P \cap \text{box}(u)$

The pair $\equiv (P_u, P_v)$

diameter of $U \equiv \text{diam}(P_u)$

which we upper bound by

\[ \Delta(u) = \begin{cases} \text{diam}(\mathcal{D}u) & \text{if } |P_u| \geq 2 \\ 0 & \text{otherwise} \end{cases} \]

\[ d(u, v) = d(\mathcal{D}u, \mathcal{D}v) \]
Def. \( W = \{(A_1, B_1), \ldots, (A_s, B_s)\} \) is a pair decomp for Pf.

1) \( A_i \cap B_i \subseteq P \)
2) \( A_i \cap B_i = \emptyset \)
3) \( \forall p, q \in P \exists i: p \in A_i \land q \in B_i \)

\( W \) is WSPD if \( \forall i \) \( A_i \cap B_i \) are \( \gamma_e \)-separated.

Thm. \( \forall 0 < \varepsilon \ll 1 \) can construct \( \varepsilon' \)-WSPD of size

\[ \frac{n}{\varepsilon d} \text{ in time } O(nh_n + \frac{n}{\varepsilon d}). \]
Proc WSPD(u, v)
If u = v and Δ(u) = 0 then return
If Δ(u) < Δ(v) swap u & v
If Δ(u) ≤ εอด(u,v) then return {u,v}
Else \{u_1, u_2, ..., u_r\} = split(u)
return U_r \bigcup \limits_{i=1}^{r} \text{WSPD}(u_i, v)

Proc WSPD(P)
1) Generate Compressed QT T
2) Return WSPD(root, root)

Lemma Let G, be an x grid, Box B, and Y ≥ X then
# box dist y from B in O((Y/X)^d)
Lemma WSPD terminates with valid WSPD.

pf Let \( p \not= q \) exist \( \exists u, v \) output st
\( p \in \mathcal{P}_u \) \& \( q \in \mathcal{P}_v \) by induction.

because \( p \not= q \rightarrow \{(p, q)\} \text{ WS} \).

Lemma We get an \( 1\varepsilon \)-WSPD.

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Lemma \# of pairs is \( O(n/\varepsilon d) \)

pf Let \((u, v)\) be a pair output.

Let \((u', v)\) be the all that generated \((u, v)\)

ie \( u' = \overline{P}(u) \)

we charge \( \overline{P}(u) \) for \((u, v)\)
Lemma Let \((u,v)\) be output pair & 
\(\bar{p}(u), \bar{p}(v)\) be parents respectively.

Then \(\max(\Delta(u), \Delta(v)) \leq \min(\Delta(\bar{p}(u)), \Delta(\bar{p}(v)))\).

\[
\begin{align*}
\Delta(\bar{p}(u)) & \geq \Delta(v) & & \Delta(\bar{p}(u)) & \geq \Delta(u) \\
& \uparrow \\
& \text{we split larger box.}
\end{align*}
\]

\(\Delta(\bar{p}(v)) \geq \Delta(u)?\)

At time we split \(\bar{p}(v)\) it was larger.
But last split was to \(\bar{p}(u)\) thus
\(\Delta(\bar{p}(v)) \geq \Delta(\bar{p}(u))\)
The 3 cases.

\[ \Delta(u') = \Delta(v) \]

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\[ \Delta(p(v)) > \Delta(u') \]
**Definition**: \( G = (V, E, w) \) is a \( t \)-spanner of \( P \subset \mathbb{R}^d \) if
\[
\| p - q \| \leq d_G(p, q) \leq t \| p - q \| \quad \forall p, q \in P.
\]

**Construction**: 1) Set \( \delta = 16 \) \( S = 3/\epsilon \)
2) Compute \( S^2 \)-WSPD \( \mathcal{Z} \)
3) For \((u, v) \in \text{Pairs}(\mathcal{Z})\) add \((\text{rep}_u, \text{rep}_v) \in E_G\).

**Theorem**: \( G \) is an \((1 + \epsilon)\)-spanner of \( P \) with \( O(n/\epsilon^2) \) edges
construction time \( O(n \log n + n/\epsilon^2) \).

**Proof**: Claim