# Triangulating a Polygon

CG
10/16/12

Recall: PSLG $\equiv$ Planar Straightline graph.

<u>Def</u> (Simple) <u>Polygonal chain</u> is a PSLG consisting of a simple cycle P.

<u>Claim</u> A Polygonal chain has a unique interior.

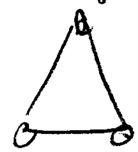
<u>Def</u> <u>Polygon</u> is Polygonal chain + interior

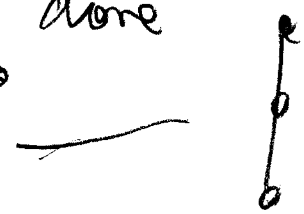
Triangulation: Addition of seg so that

1) Still PSLG
2) Interior is decomposed into triangles

<u>Thm</u> Every simple Polygon can be triangulated.

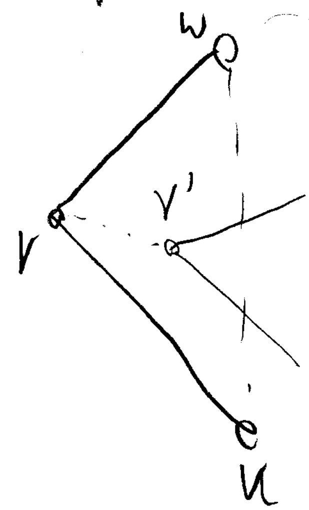
Pf Induct on # of seg or points $n$

base case: $n=3$  done

Assume no $180^\circ$ angles 

$n > 3$ true for $m < n$

Let $v$ be left most point with neigs $w$ & $u$




1) Case 1 seg $= [w,u]$ is interior

We get two Poly $|P_1| = 3$ $|P_2| = n-1$

2) Case 2 $\exists$ point $v'$ interior to Tri $= [w,v,u]$

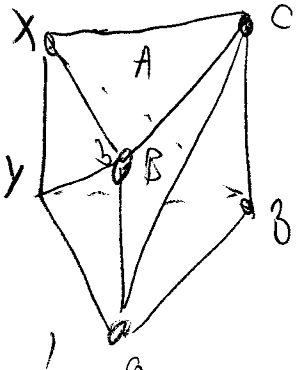
Let $v'$ be left most such point

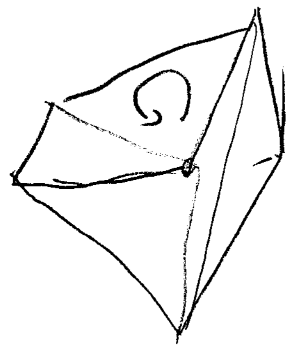
Seg $= [v,v']$ is inter

$|P_1| < n$ & $|P_2| < n$

Thm Not Every simple polygonal surface (Polytope) in 3D can be decomposed in Tetrahedra

eg prism



By Contra!
Consider Tet with face B

Missing vertex is X or Y not 3

not X since seg [a, X] is outside.
not Y since seg [3, Y] is outside.

In General: Test if polygonal surface is decomposible is NP-Hard.

# Guarding A Polygon

Input: Polygon $P$

Output: Locations $p_1 \cdots p_k \in P$ (guards)

1) Guards cover $P$
2) $k$ small.

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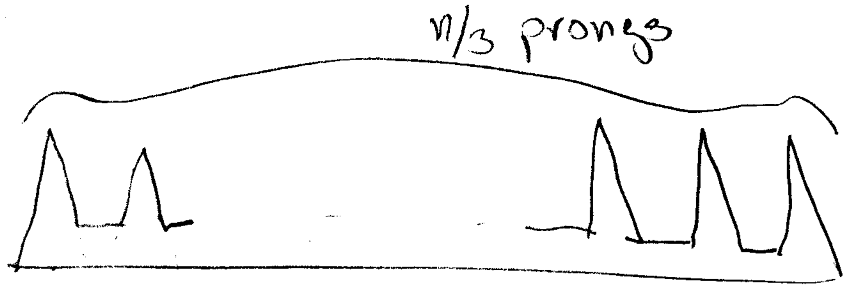
<u>Thm</u> A polygon $P$ with $n$ vertices $\lfloor n/3 \rfloor$ guards suffice and maybe necessary.

Necessary: $P =$




$|P| = n$ Needs a guard per prong.

# $n/3$-guards Alg$(P)$

1) Tri $P$ $\bar{P}$
2) 3-color $\bar{P}$
   a) Construct geometric dual $T$ (a degree 3 tree)
   b) 3-Color $\bar{P}$ by traversing tri's in an inorder fashion.
3) Pick least used color.

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Only non-linear time step is 1)

2D-algorithm

Proof $\Rightarrow O(n^2)$

Known: $O(n)$ Chazelle

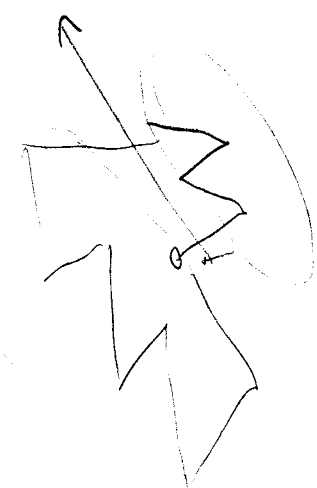
Today: $O(n \log n)$ (sweep line)

This Class: $O(n \log^* n)$ Seidel (incremental randomized)

Def $\log^* n = \min_K \underbrace{\log \log \cdots \log}_{K} n \leq 1$

Prob: Give a $O(n)$ time alg to determine which side of an edge is interior/exterior of a simple poly)

Prob: test $p \in Int(P)$ in $O(n)$ time.



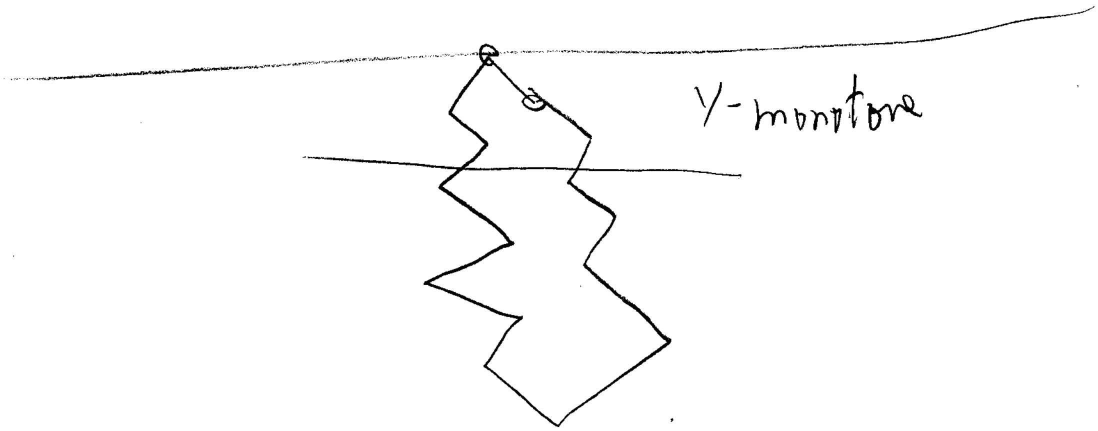
3.14 $O(n \log n)$ OK

$O(n)$ ?

Trap $\Rightarrow$ Tri

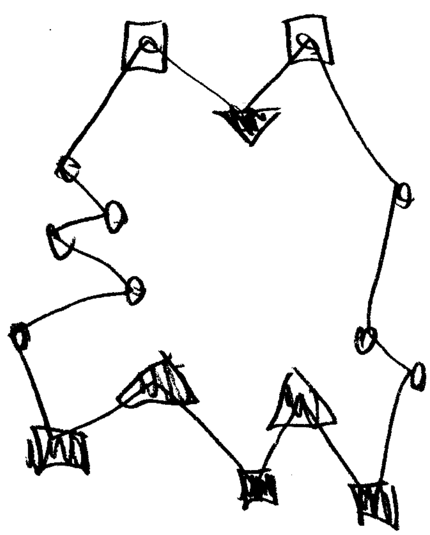
## Step 1 Partition into Monotone Polygons

**Def** P is Y-monotone if
Every horizontal line $l$
$l \cap P$ is connected or empty




Alg type: Line Sweep $O(n \lg n)$ time

**Def**



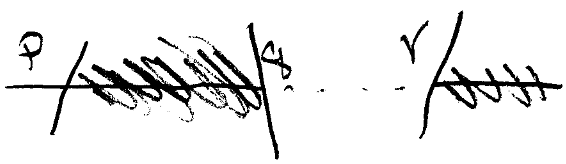
□ = start vertex
■ = end
◉ = reg
▲ split
▼ merge

<u>Claim</u> P is y-monotone iff no split or merge vertices.

($\Rightarrow$) (easy)

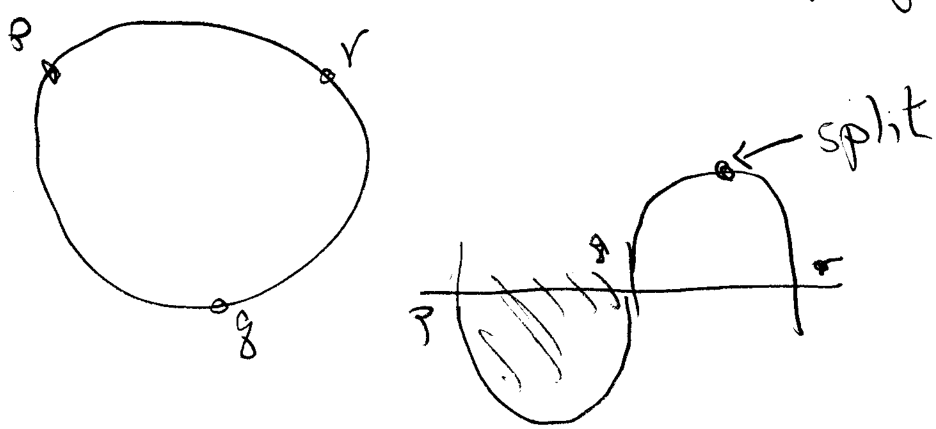
($\Leftarrow$) (not mono $\Rightarrow$ $\exists$ split or merge)

assume not mono




<u>Case 1</u>


<u>Case 2</u>



# Alg Sweep Line (top-to-bottom)

Events: endpoints

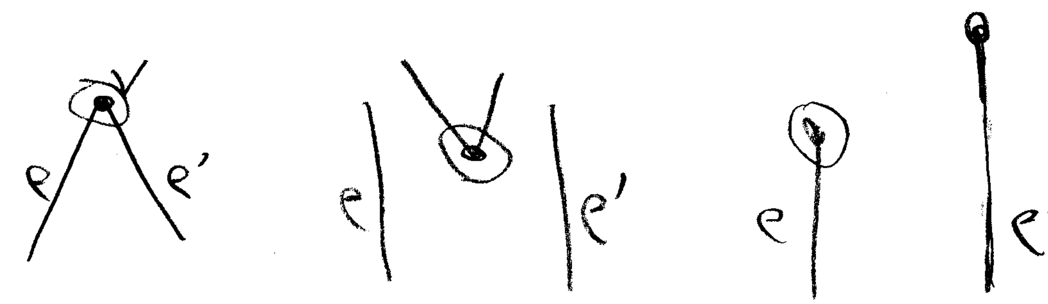
Dictionary: Intervals (sorted)

Interval: (left-seg, right-seg, helper vertex)

vertex: 2 edges before & after

Def helper(e, e') ≡ lowest vertex above ℓ
and between e & e'.

eg




no horizontal seg.

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Procedure add(p, q) ≡ add seg from p to q if [p, q] not already an edge.

# Make Monotone(q : event)

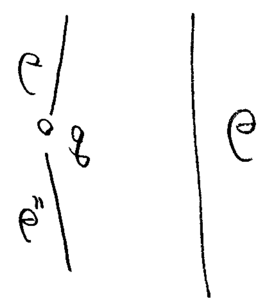
Case (Start Vertex)

1) Add new interval
2) set helper ⇐ $q_e$

Case (End Vertex)

1) if helper is a merge vertex then add (q, helper)
2) remove interval.

Case (Regular)




1) add (q, helper)
2) replace e with e''
3) helper ⇐ q

Case (Split)

1) add (q, helper)
2) "split" interval say $I_1, I_2$
3) helper($I_1$) = helper($I_2$) ⇐ q

Case (Merge)

1) add(help$_{PL}$, 8); add(help$_{PR}$, 8)

2) "Merge" intervals

3) help $\leftarrow$ 8

# Another View

1) Make Trapezoidal Decom (sweep line)
2) For each trap add a diagonal if possible
3) types of Traps