CG 10

12/4/12

de Casteljau Alg, Bezier Curves
Bernstein Polys

eg

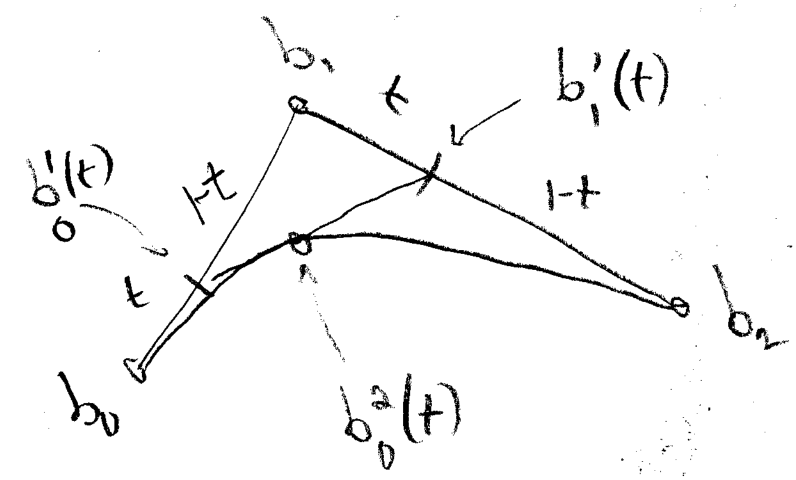
Let $b_0, b_1, b_2 \in R^2$ & $0 \le t \le 1$

$$b_0^1(t) = (1-t)\,b_0 + t\,b_0$$

$$b_1^1(t) = (1-t)\,b_1 + t\,b_2$$

$$b_0^2(t) = (1-t)\,b_0^1(t) + t\,b_1^1(t)$$

picture



As a Polynomial

$$b_0^2(t) = (1-t)\big((1-t)b_0 + t\,b_1\big) + t\big((1-t)b_1 + t\,b_2\big)$$

$$= \underbrace{(1-t)^2}_{\alpha} b_0 + \underbrace{2t(1-t)}_{\beta} b_1 + \underbrace{t^2}_{\gamma} b_2$$

$$\alpha + \beta + \gamma = \big((1-t) + t\big)^2 = 1$$

de Castaljan Alg

$b_0 \cdots b_n \in \mathbb{R}^d \quad t \in \mathbb{R}$

set $\quad b_i^r = (1-t)\, b_i^{r-1}(t) + t\, b_{i+1}^{r-1}(t) \qquad \begin{array}{l} r = 1, \cdots, n \\ i = 0, \cdots n-r \end{array}$

b_i^r is poly curve of degree r

<u>Claim</u> de Casteljau commutes with affine maps

ie $x \to Ax+b$

$$\begin{tikzcd} \cdot \arrow[r, "Af"] \arrow[d, "Cast"'] & \cdot \arrow[d, "Cast"] \\ \cdot \arrow[r, "Af"] & \cdot \end{tikzcd}$$

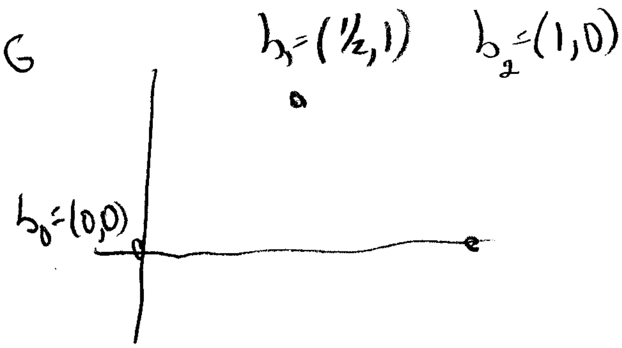
$$b_0^1(t) = (1-t)b_0 + tb_1$$

$$= (1-t)(Ab_0 + b) + t(Ab_1 + b)$$

$$= A((1-t)(b_0)) + (1-t)b + Atb_1 + tb)$$

$$A((1-t)b_0 + tb_1) + b$$

WLOG

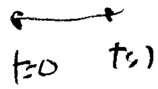


Representing Curves
Implicit & Parametric

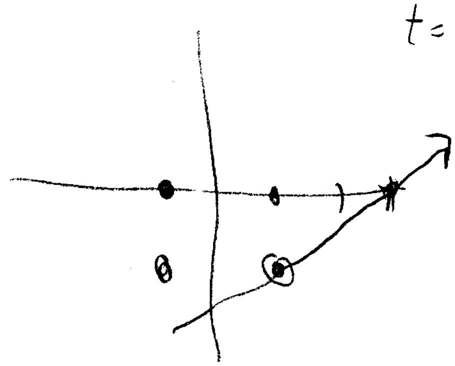
$t=0 \quad (1,-1)$

$t=1 \quad (3,0)$

Parametric



Line $\to \mathbb{R}^2$



$x = 2t+1$

$y = t-1$

Implicit	Parametric
$x^2+y^2-1=0$	$x = \cos\theta$ $y = \sin\theta$

Implicit $ax^2+bxy+cy^2+dx+ey+f=0$

Parabolas $y^2 = 4ax$

Ellipses $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ Circles $a=b$

Hyperbolas $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

Polynomial Curves of degree 1 & 2

Degree 1

$$x(t) = a_1 t + a_0$$

$$y(t) = b_1 t + b_0$$

$a_1 \cdot b_1 = 0$ then curve is horizontal or vertical line

$$b_1 x = a_1 b_1 t + a_0 b_1$$

$$a_1 y = a_1 b_1 t + a_1 b_0$$

$$\boxed{b_1 x - a_0 b_1 = a_1 y - a_1 b_0}$$ linear implicit

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Degree 2

$$x(t) = F_1(t) = a_2 t^2 + a_1 t + a_0$$

$$y(t) = F_2(t) = b_2 t^2 + b_1 t + a_0$$

$a_2 \neq 0$ or $b_2 \neq 0$ $\quad \rho = \sqrt{a_2^2 + b_2^2}$

$$R = \begin{pmatrix} b_2/\rho & -a_2/\rho \\ \frac{a_2}{\rho} & b_2/\rho \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix}$$ change of varibls

$$x_1(t) = \left(\frac{b_2 a_2}{\rho} t^2 + \cdots - \frac{a_2 b_2}{\rho} t^2 + \cdots \right)$$

$$x_1(t) = a_1' t + a_0'$$

$$y_1(t) = b_2' t^2 + b_1' t + b_0' \quad b_2' \neq 0$$

$a_1' = 0$ degenerate

goal eliminate b_1'

$$u = t + \frac{b_1'}{2b_2'}$$

Our form is

$$x(u) = a_1 u + a_0$$

$$y(u) = b_2 u^2 + b_0$$

a translation

$$x(u) = a u$$

$$y(u) = b u^2 \quad b > 0$$

$$\frac{b}{a^2}x^2 = bu^2 = y$$

<u>Thm</u> Parametric Quad are parabolas

<u>no</u> Ellipses or Hyperbolas

Computing Bezier by subdivision

fix $b_0 \cdots b_n$

$$\begin{array}{ccccc} b_0 & & & & \\ \vdots & b_0^1 & & & \\ \vdots & \vdots & b_0^2 & & \\ \vdots & \vdots & \vdots & \ddots & \\ b_n & b_{n-1}^1 & b_{n-2}^2 & & b_0^n \end{array} \qquad b_i^j \in \mathbb{R}^d$$

two control polygons (polygonal paths)

$$(b_0 \, b_0^1 \cdots b_0^n) \quad (b_0^n \, b_{n-1}^{n-1} \cdots b_n^0)$$

<u>Claim</u> $\underset{0 \le s \le 1}{(b_0 \cdots b_n)} \equiv \underset{\substack{0 \le s \le t \\ 0 \le s/t \le 1}}{(b_0^1 \cdots b_0^n)} * \underset{\substack{t \le s \le 1 \\ 0 \le \frac{s-t}{1-t} \le 1}}{(b_0^n \cdots b_n^0)}$

all give the same curve.

<u>Menelao's Thm</u>

$$b[0,t] = (1-t)\,b_0 + t\,b_1$$

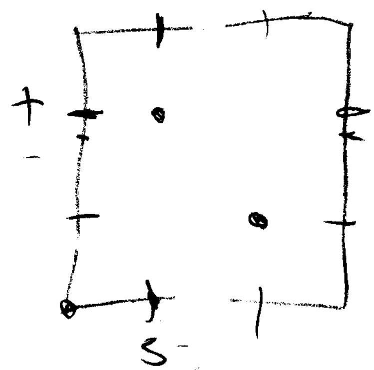
$$b[s,0] = (1-s)\,b_0 + s\,b_1$$

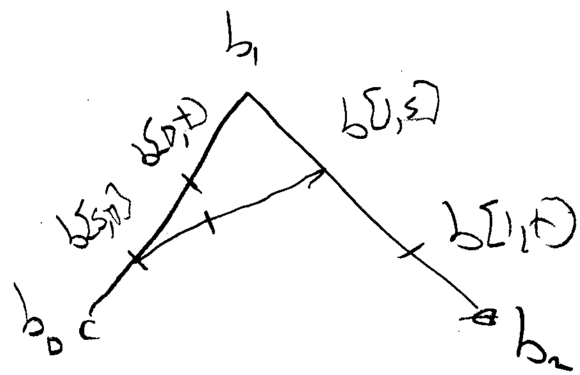
$$b[1,t] = (1-t)\,b_1 + t\,b_2$$

$$b[s,1] = (1-s)\,b_1 + s\,b_2$$

$$b[s,t] = (1-t)\,b[s,0] + t\,b[s,1]$$

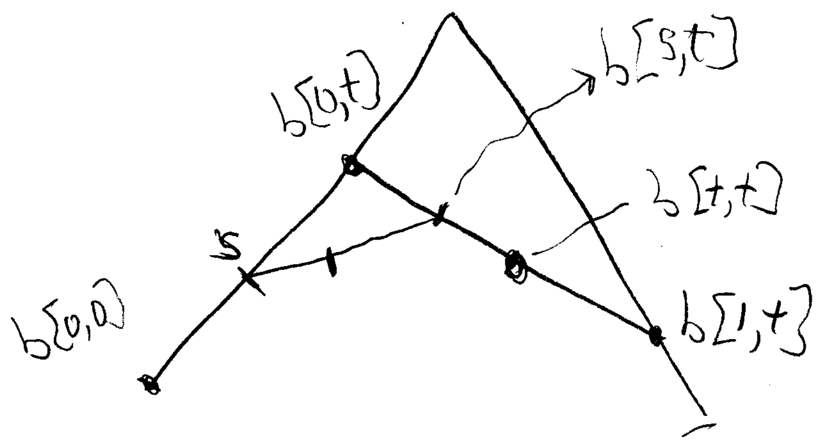
$$b[t,s] = (1-s)\,b[0,t] + s\,b[1,t]$$





<u>Thm</u> $b[s,t] = b[t,s]$

<u>pf</u> by expansion



$$
\begin{array}{lll}
b[0,0] & b' & \\
 & \shortparallel & \\
b[0,t] & b[0,s] & \\
b[t,t] & b[s,t] & b[s,s] \\
 & \shortparallel & \\
 & b[t,s] &
\end{array}
$$

$$b' = \frac{(1-s)}{t} b[0,0] + \frac{s}{t}[b[0,t]]$$

$$= b[0,s]$$

<u>Fact</u> $\alpha b[r,t_2] + \beta b[s,t_2] = b[\alpha r + \beta s, t_2] \quad \alpha + \beta = 1$

Bernstein Polynomials

Bernstein: $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$

$$\binom{n}{i} = \begin{cases} \dfrac{n!}{i!\,(n-i)!} & 0 \le i \le n \\ 0 & \text{o.w.} \end{cases}$$

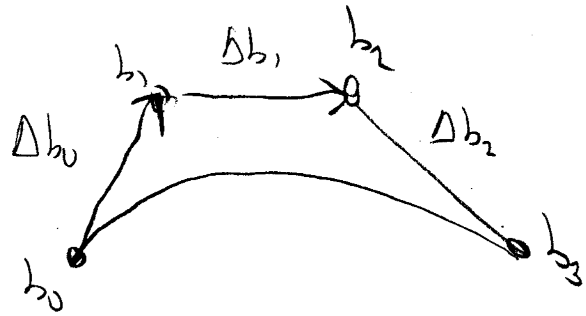
$n=2$

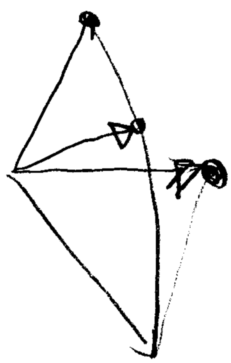
$B_0^2 = \binom{2}{0}(1-t)^2 \qquad B_1^2 = \binom{2}{1} t(1-t) \qquad B_2^2 = \binom{2}{2} t^2$

$= 1-2t+t^2 \qquad 2t-2t^2 \qquad t^2$

$\sum B_i^2 = 1$

Derivative of a Bezier Curve





vector curve $(\Delta b_0, \Delta b_1, \Delta b_2)$

$\Delta b(t)$

$$\frac{\partial b(0)}{\partial t} = d\,\Delta b_0 \qquad d = \text{degree}$$