Mesh Generation

Goal: Partition domain into simplices

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Simplexes: vertex, segment, triangle, tetrahedron

Partition: Intersection of 2 simplices in a simplex

Conforming to input

Well-shaped simplices

a) no small angles < 0°

b) no large angles > 80°

Small number of simplices (nominal size)
Aspect Ratio

\[ A(a, b, c) = \frac{\text{longest-side}}{\text{alt}} \]

\[ R(a, b, c) = \frac{\text{longest-side}}{\text{shortest-side}} \]

\[ \frac{1}{\text{Smallest-angle}} \]

\[ \frac{1}{180^\circ - \text{largest-angle}} \]

radius-edge ratio = \( \frac{\Gamma}{e} \)

\( \Gamma = \text{radius of circum sphere} \)

\( e = \text{shortest edge} \)
Mesh Generation Methods

1) Quadtree (today)
2) Delaunay Refinement (Thursday)
3) Advancing Front
4) Ball-Packing
5) Voronoi Refinement

In 2D our input will be PSLG.
Simplex & Simplical Complex

Def: \( p_0, \ldots, p_k \in \mathbb{R}^d \) are \underline{affinely independent} of dimension \( k \) if \( p_i - p_0, \ldots, p_k - p_0 \) are independent.

Def: If \( p_0, \ldots, p_k \) are a-ind then \( \text{CC}(p_0, \ldots, p_k) \) is a \underline{k-simplex} & \( \forall S \subseteq \{p_0, \ldots, p_k\} \) \( \text{CC}(S) \) is a \underline{sub-simplex}.

Def: A set \( K \) of simplices in \( \mathbb{R}^d \) is a \underline{Simplicial Complex} if:

1) \( K \) is closed under sub-simplex
2) \( S, t \in K \) then \( S \cup t \) a sub-simplex of \( K \)

Def: \( \dim(K) = \max \{ \dim S : S \in K \} \)
Note PSLG is a 1-dim simplicial complex in $\mathbb{R}^2$

$K$ & $K'$ are simplicial complexes

**Def** $K'$ is a refinement of $K$ if

$\forall s \in K$ of dim $k$, $\exists s_1, \ldots, s_k \in K'$ of dim $k$

$s.t.$ $s = \bigcup_{i=1}^{k} s_i$

**Input** Simplicial complex $K$ & Domain $\mathcal{N}$

$\forall s \in K \implies s \subseteq \mathcal{N}$

**Output** refinement $K'$ of $K$ s.t.

$\forall s \in \mathcal{N} \ni \bigcup_{s \in K'} s = \mathcal{N}$
Quad-Tree Meshing

Input: set $X \subseteq \mathbb{R}^2$ of points $X \subseteq B$ (box) $|X| = n$

Def: $QT$ is a tree of nested square boxes. The children of box $b$ are either

1) empty (leaf box)
2) 4 children of half the size (split of $b$)

Neighbors: 4 direct neighbors
8 extended neighbors
Def: A tree is balanced if every leaf box has no side containing more than one interior node.
Build-QT \((X, B)\)

Init: QT \(T = (X, B)\)

1) While each box \((x', b)\) at \(b\) in "crowded"
   split \(b\) and assign \(x'\) to new boxes.

2) Balance \(T\) by splitting.

3) Split all boxes containing a point until it
   has 8 extended neighbors (leaf boxes).
Def: Let box \( b \) be crowded if \( \exists x \in b \) and one of the following holds:

1. \( \exists y \neq x \in b \)
2. \( \exists y \in X \) s.t. \( \text{dist}(x, y) \leq 2\sqrt{2} \cdot \text{side length}(b) \)
3. An extended neighbor of \( b \) in split.

Warping

\( x \in b \) & \( y \) closest corner of \( b \) to warp \( y \).

\[ \begin{array}{c}
\begin{array}{c}
X_0 \quad \text{Warp} \quad \Delta \quad \Rightarrow \\
\end{array}
\end{array} \]

\( b \) empty & not warped

\[ \begin{array}{c}
\begin{array}{c}
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\end{array}
\end{array} \]
Thm \( T \) is a QT & \( T' \) is its balanced version then \( |T'| = O(|T|) \), \( |T'| \) = # boxes

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Note \( T \) is a proper \( k \)-ary tree

Note A proper \( k \)-ary tree with \( i \) internal nodes has size \( ki + 1 \) (induct)

boxes of \( T \) are old

Claim A new internal box has an extended neigh which is old

proof by contraction
Let $b$ be the smallest internal new box with no old ext neigh. 

$b$ internal ⇒ a side of $b$ is split twice.

eg

\[ b \rightarrow b' \]

$b'$ is new with no old neigh. Contradition!

\[
\# \text{int}(T') \leq 8 \cdot n \quad (9n)?
\]

\[
\#(T') \leq 4 \cdot \#(\text{int}(T') + 1) \leq 32n + 1
\]

Thm: Balanced QT can be computed in $O(dn)$ time $d = \text{depth}$. 