CG
10/30/12

Lower Bound on Mesh Size

Suppose M is an optimal size mesh for PSLG G with Bounding Box B.

radius-edge $\leq \rho$

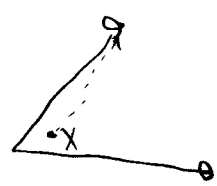
<u>Question</u>: Bd $|M|$ from below.

Feature $\equiv$ vertex, edge, face

$f, g \in$ Feature then f & g are <u>disjoint</u> if $f \cap g = \emptyset$

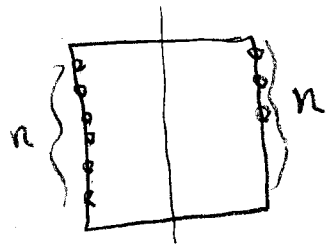
<u>Def</u> (Local feature Size) $x \in R^2$ then

$lfs(x)$ = distance from x to 2nd nearest feature (G PSLG)



Goal: (Thm) $|M| = \Theta\left(\frac{1}{\alpha}\int_{x\in B} 1/\mathrm{lfs}^2(x)\right)$

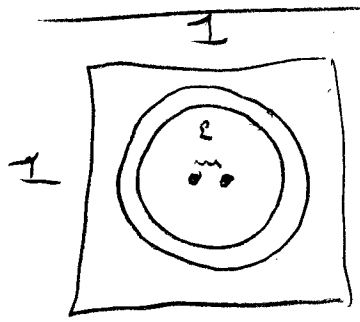
eg:



$\equiv$ PSLG

$$\int_{x\in B}\frac{1}{\mathrm{lfs}^2(x)} = 2\int_{x\in B/2}\frac{1}{\mathrm{lfs}^2(x)} = 2\int_0^1\int_0^{1/2}\frac{1}{(x+1/n)^2}\,dx\,dy \approx 2\int_{1/n}^{1/2}\frac{1}{x^2}\,dx$$

$$= 2\left[1/x\right]_{1/2}^{1/n} = 2(n-2) \approx 2n$$



$$\int_\varepsilon^1 \frac{2\pi r}{r^2}\,dr = 2\pi\log r\Big]_\varepsilon^1 \approx -\log\varepsilon = \log 1/\varepsilon$$

<u>To prove thm we go to the dual</u>

Suppose V_x is Voronoi cell centered at x

<u>Def</u> R = radius of smallest ball containing V_x centered at x
r = " " largest " contained in " "

<u>Def</u> Aspect-Ratio of $V_x = R/r$

Let $P \subseteq \mathbb{R}^d$ $\quad$ $Del(P)$ = Delaunay of P
$Vor(P)$ = Voronoi of P

<u>Thm</u> If $Vor(P)$ has aspect-ratio $\leq \rho$ then
$Del(P)$ has radius-edge $\leq \rho/2$

<u>pf</u> $\rho/2 = R/2r \geq R/|e| =$ radius-edge

<u>Thm</u> Del(P) has radius-edge $\le \rho$ then Vor(P) has aspect-ratio $\le C_d \rho$ where C_d depends only on dim d.

<u>pf</u> See handout Thm 3.2
2D case in paper by Mitchell

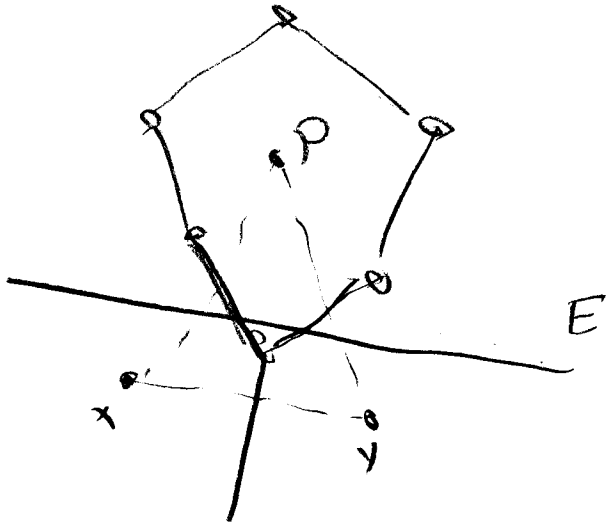
Suppose Input: PSLG G Box
Set of points P st
1) Del(P) is a refinement of G
2) Vor(P) has aspect-ratio $\le \rho \ge 2$

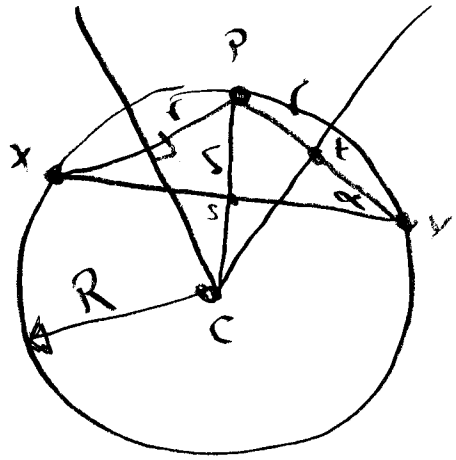
<u>Claim</u> $p \in P$ $lfs(p) \ge 2r/\rho$ r = inradius of V_p

<u>pf</u> Let F be a feature of G (vertex or edge)

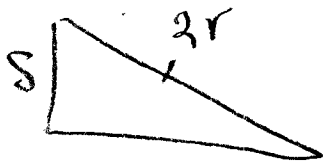
<u>Case 1</u> F is a vertex Then $F \notin B(p,r)$.

Case 2 F is an edge E.
It may be $E \cap B(p,r) \ne \emptyset$





Tri 



congruent

$$\rho = \frac{R}{r} = \frac{2r}{\delta} \Rightarrow \delta \geq \frac{2r}{\rho}$$

<u>Claim</u> $\forall x \in V_p \quad lfs(p) \geq r/\rho$

<u>proof needed</u>

<u>Lemma</u> $\displaystyle\int_{V \in V_p} \frac{1}{lfs^2(v)}\, dA \leq \pi \rho^4$

$$\int_{V \in V_x} \frac{1}{lfs^2(v)}\, dA \leq |B(p, R)| \cdot \frac{1}{(r/\rho)^2} = \frac{\pi R^2}{r^2}\rho^2 = \pi \rho^4$$

<u>Thm</u> $\displaystyle |M| = \Omega\left(\int_{x \in Box} \frac{1}{lfs^2(x)}\right)$

$$\int_{x \in Box} \frac{1}{lfs^2(x)} \leq \sum_{V_p} \int_{x \in V_p} \frac{1}{lfs^2(x)}\, dA \doteq |M|\, \pi \rho^4$$

$$|M| \geq \frac{1}{\pi \rho^4} \int \frac{1}{lfs^2(x)}$$

3.1 Bounded Radius-Edge Ratio

Definition 3.1 (Bounded radius-edge ratio) *A DT has* **radius-edge ratio** *bounded by $\rho \geq 1$ if for each Delaunay simplex the ratio of its circumscribed sphere radius to the smallest edge is bounded by ρ.*

In 2D, if a DT has radius-edge ratio bounded by ρ then its smallest angle is at least $sin^{-1}(1/(2\rho))$. Thus bounded radius-edge ratio implies bounded aspect ratio and vice verse. In 3D and higher, the bounded radius-edge ratio does not guarantee that the minimal dihedral angle is bounded. A notorious example is a sliver in 3D which is a simplex whose four nodes are placed almost in a square along the equator of their circumscribing Delaunay sphere. The radius-edge ratio in that case is about $\sqrt{2}$, but the area of the sliver can be arbitrarily close to zero. Thus, the radius-edge ratio condition is weaker than the aspect-ratio condition and hence all the structure theorems and algorithms presented in this paper apply to the DT with bounded aspect ratio.

3.2 Density of Delaunay Diagrams

Here we show that if $DT(P)$ has a bounded radius-edge ratio, then its 1-dimensional skeleton is a density graph, and hence has a bounded degree and a small sphere separator. We will use this result in section 5 to develop an $O(\log n)$ parallel time n processor parallel algorithm for constructing the Delaunay diagram. In all the lemmas and theorems in this section, let P be a point set in $\mathbf{R}^d$ such that $DT(P)$ has ratio bounded by $\rho > 1$.

Theorem 3.2 (Density Embedding) *P is an α-density embedding of $DT(P)$, where α is a constant dependent only on the dimension d and ρ.*

The standard volume argument can not be used to prove Theorem 3.2 because of slivers. We use the following notation in our proof: for each point $p \in P$, let $N(p)$ be the set of all Delaunay simplices incident to p. For each Delaunay simplex $T \in N(p)$, we refer to the vector from p to the center of the Delaunay sphere of T as the *radius vector* of T. Two simplices are *neighboring* if they share a common edge. The following lemmas will be used to prove Theorem 3.2.

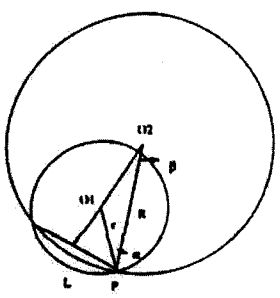


Figure 2: Projection of two intersecting spheres on the plane defined by their radius vectors from P.

Lemma 3.3 *For $\theta = arcsine(1/2\rho)/4$ there is a constant ρ_1 dependent only on ρ, such that for all $p \in P$, for each pair of Delaunay simplices T_1 and T_2 in $N(p)$, with radii R and r, if the angle between the two radius vectors is smaller than θ, then $\frac{R}{r} \leq \rho_1$.*

Proof: If $R \leq r$ the Lemma is obvious, therefore assume $R \geq r$. We depict the case in Figure 2, where we assume $\alpha \leq \theta$. We have $R\sin(\beta) = r\sin(\alpha+\beta)$, so $\frac{R}{r} \leq \frac{\sin(\alpha+\beta)}{\sin(\beta)} \leq \frac{1}{\sin\beta}$. The vertices of the simplex of the smaller Delaunay sphere can not be in the interior of the larger Delaunay sphere, so there is an edge whose size is less then $2r\sin(\beta+\alpha)$. The bounded radius-edge ratio property implies $r/(2r\sin(\beta+\alpha)) \leq \rho$. Hence $\sin(\beta+\alpha) \geq 1/2\rho$ and therefore $\beta+\alpha \geq arcsine(1/2\rho)$ and $\beta \geq 3arcsine(1/2\rho)/4$. Assign $\rho_1 = 1/\sin[3arcsine(1/\rho)/4]$ to get $R/r \leq \rho_1$. □

Lemma 3.4 *Let T_1, T_2 be two simplices, $E(e)$ an edge of $T_1(T_2)$, $R(r)$ the circumscribing sphere radius of $T_1(T_2)$.*

1. *If T1 and T2 are neighbors then $|E|/|e| \leq 4\rho^2$.*
2. *If $R/r \leq \rho_1$ then $|E|/|e| \leq 2\rho\rho_1$.*

Proof:

1. If ge is an edge common to the two simplices, then $|E|/|e| = (|E|/|ge|)(|ge|/|e|) \leq 4\rho^2$.
2. $|E|/|e| = (|E|/R)(R/r)(r/|e|) \leq \rho\rho_1 2$ □

Let $\gamma = \max(2\rho\rho_1, 4\rho^2)$. To show the DT is a density graph, we cover a very small sphere S centered at a point $p \in P$ by a collection of circular patches with cone angle θ. The following lemma is a folklore.

Lemma 3.5 *There is a constant K dependent only on θ and d such that there is a cover of the unit sphere in $\mathbf{R}^d$ with no more than K circular patches whose angle is equal to θ.*

Proof of Theorem 3.2. Let S be a very small sphere centered at $p \in P$. We cover S according to Lemma 3.5. Each radius vector from p intersects sphere S in at least one cone patch (the patches are not necessarily disjoint, so it could intersect more than one patch). Assign to each radius vector a label which corresponds to one of the patches it intersects. If two radius vectors have the same label, then by Lemmas 3.4 and 3.3, the maximal ratio of the edges belonging to the two simplices is bounded by γ.

Let e and E be the shortest and the longest Delaunay edges, respectively, incident to p. There is a path between e and E through edges that belong to neighboring simplices incident to p. In each transition of the path, the edge lengths can grow by at most a factor of γ.

We assign a label to each edge in the path. The label indicates the patch that the edge's radius vector intersects. If a label appears more than once in the path, we can "erase" all labels between last and first appearance of the label, and instead use the ratio information forced by the label, which is γ. This "erasing" process reduces the number of labels to less then K because no label can repeat. Therefore the ratio of $|E|$ to $|e|$ is bounded by $\alpha = \gamma^{2K}$ and hence P is an α-density embedding of $DT(P)$. □

Lemma 3.6 *The vertex degree of each node in an α-density graph in d dimension is bounded by $(2\alpha^2 + 2\alpha)^d$*

Proof: For each $p \in P$, the neighboring nodes of p are contained in the sphere with radius $\alpha|e|$ centered at p, where e is the smallest edge incident to p. Let q be one of p's neighbors, then q has an edge of length at least $|e|$, so q's nearest neighbor is no closer than $|e|/\alpha$. Therefore, the sphere centered at q of radius $|e|/(2\alpha)$ does not intersect with the sphere centered at any other neighboring node of p of radius $|e|/(2\alpha)$. A simple volume argument gives the bound. □