Delaunay Refinement
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2D only

Input: [PSLG] all angles > 60° (G)

Output: 1) 2D simplicial complex
2) A refinement of the PSLG
3) Delaunay
4) no small angle
5) constant times opt size

Def \( \text{dfs}(x) = \text{dist to 2nd nearest disjoint feature} \)

Def \( f: \mathbb{R}^d \to \mathbb{R} \) is \( \alpha \)-Lipschitz if \( \forall p, q \in \mathbb{R}^d \)

\[ |f(p) - f(q)| \leq \alpha \text{ dist}(p, q) \]

Claim \( \text{dfs} \) is 1-Lipschitz i.e.

\[ \text{dfs}(p) \leq \text{dfs}(q) + \text{dist}(p, q) \]
Algorithm

**Def** Circumball of a simplex is min radius ball $B$ with vertices on $B$.

**Def** $P$ encroaches simplex $S$ if $P \in \text{Inter(Circumball}(S))$

**Def** $S'$ encroaches on $S$ if Circumcenter($S'$) $\in \text{Inter(Circumball}(S))$

**Def** Segment is a sub-segment of an edge of $G$.

**Delaunay Refinement ($G$) (Overview)**

1. Add a bounding box to $G$
2. Compute Delaunay of $V(G)+\text{Box}$
3. While
   1. A segment is encroached add circum-center
   2. A $\Delta$ is skinny add circum-center.
Subroutine: Split(segs, S)

1) Add circumcenter of s to V & update DT(V)
2) remove s from S
3) add halves of s to S.
Delaunay Refinement \((PSLG \ G, \ angle \ \alpha \ or \ radius-edge \ \rho)\)

Init
1) Add bounding box to \(G\)
2) \(S = \text{edges}(G)\)
3) \(V = \text{vertices}(G)\)
4) \(T = \text{DT}(V)\)

While \(\exists\ \text{encroached seg on skinny tri}\) do
1) While \(\exists\ \text{seg (s encroached)}\) split seg(s)
2) If t skinny (radius-edge \(\rho\)) then do \((\star)\)
\[\text{If t encroaches a seg S then do} \]
\[\text{split seg}(S)\]
\[\text{else split tri}(t)\]

Return \(\text{DT}(V)\)
Def \[ NN_t(P) = \text{nearest vertex in } V \text{ at last time } P \text{ was considered for insertion before } t. \]

E.g. at step (x) \( P_3 \) CircumCenter(+) was considered but may not have been added.

Def Containing Dimension of \( P \) = \( \min \text{ dim feature containing } P. \)

E.g. \( P \) is an input point then \( CD(P) = 0 \)

- \( P \) interior to an edge \( CD(P) = 1 \)
- \( P \) w/o \( CD(P) = 2 \)

Lemma \( \exists \) constants \( c_e \& c_t \) depending only on \( a \) s.t. \( \forall t \)

1) If \( CD(P) = 0 \) then \( lfs(P) \leq NN_t(P) \)
2) If \( CD(P) = 1 \) then \( lfs(P) \leq c_e NN_t(P) \)
3) If \( CD(P) = 2 \) then \( lfs(P) \leq c_t NN_t(P) \)
Proof: Induction on execution time, $t$.
Assume $t$ and show $t+1$

**Case $CD(p) = 0$** ($p$ will only be considered once)

$$
lfs_G(p) \leq lfs_V(p) \leq NN(p) = NN_{t+1}(p)
$$

**Case $CD(p) = 1$**

Let $p \in E_p$ (input edge) & $p$ = circum center of segment $S_p$

If $p \in S_p \subset E_p$

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$S_p$ must have been encroached by some point $a$. Pick $a \in Ball(S_p)$ as follows.

1) $a \in Ball(S_p)$ at time $splitseg(S_p)$
   set $a$ to closest such point to $p$. 

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Else let $a \in B(S_p)$ be circumcenter of skinny tri that yielded to $p$.

**Subcase** $CD(a) = 0$

\[ \text{let } a \in B(S_p) \text{ be circumcenter of skinny triangle that yielded to } p. \]

\[ a \text{ was closest point.} \]

\[ \text{we need } \boxed{1 \leq C_e} \]

**Subcase** $CD(a) = 1$

Let $a \in E_a$ (input edge)

(lets assume input angles $\geq 90^\circ$)

Thus $E_a \cap E_p = \emptyset$

\[ d_e(p) \leq \text{dist}(p, E_a) \leq \text{dist}(p, a) = NN_{t+1}(p) \]

\[ \boxed{1 \leq C_e} \]

(60$^\circ$ case?)
Subcase $CD(a) = 2$

By induction $\ell f_{5}(a) \leq C_{t} NN_{t}(a) \leq C_{t} r'$

1) $NN_{t}(p) = r$ (will yield to $S_{p}$)

2) $a \in B(p, r) \land x, y \in B(a, r') \Rightarrow r' \leq \sqrt{2} r$

$\ell f_{5}(p) \leq \ell f_{5}(a) + \text{dist}(p, a) \leq C_{t} r' + r$

$\leq C_{t} \sqrt{2} r + r$

$\leq (\sqrt{2} C_{t} + 1) NN_{t+h}(p)$

We need $1 + \sqrt{2} C_{t} \leq C_{e}$
Case $CD(p) = 2$

WLOG $a$ added before $b$.

Subcase $CD(b) = 0$

$\text{lfs}(p) = r = \text{NN}_{\text{th}}(p)$

Subcase $CD(b) = 1$

(induct) $\text{lfs}(b) \leq C_e \text{NN}_t(b) \leq C_e \text{dist}(b, a)$

radio-edge $p \leq \frac{r}{\rho}$ i.e. $e \leq \frac{r}{\rho}$, $\bar{p} = \frac{\rho}{e}$

$\text{lfs}(p) \leq \text{lfs}(b) + \text{dist}(p, b) = \text{lfs}(b) + r$

$\leq C_e \text{dist}(b, a) + r$

$\leq C_e \bar{p} r + r = (C_e \bar{p} + 1) r$

$= (C_e \bar{p} + 1) \text{NN}_{\text{th}}(p)$

need $(C_e \bar{p} + 1) \leq C_t$
Subcase, CD(b) = 2

Same as last case but $C_e$ is $C_t$

ie $(C_t \bar{\rho} + 1) \leq C_t$

\[
\frac{1}{1-\bar{\rho}} \leq C_t \quad \text{or} \quad \frac{\bar{\rho}}{\rho-1} \leq C_t
\]

Our list of needed conditions for $\rho$

\[
1 \leq C_e \\
\frac{\rho}{\rho-1} \leq C_t \\
1 + \sqrt{2} C_t \leq C_e \\
1 + \rho C_e \leq C_t
\]
\[ c_t = \frac{c_e}{\sqrt{\lambda}} - \frac{1}{\sqrt{\lambda}} \text{ slope } \frac{1}{\sqrt{\lambda}} \Rightarrow \bar{\rho} \leq \frac{1}{\sqrt{\lambda}} \]

\[ c_t = \bar{\rho} c_e + 1 \text{ slope } \bar{\rho} \quad \text{on} \quad \rho = \sqrt{\lambda} \]
\[ p = \frac{r}{2} \geq \sqrt{2} \]

\[ \frac{e}{r} = \sin \alpha \]

\[ \left( \frac{1}{2 \sqrt{2}} \right) = \sin \alpha \quad \Rightarrow \quad \alpha \approx \sin^{-1} \left( \frac{1}{2 \sqrt{2}} \right) \approx 21^0 \]
If $p \in \text{Output}$ then $d_{fs}(p) \leq (c_{e+1}) \text{NN}_{\text{output}}(p)$

**Proof**

Let $g$ be NN of $p$ in output i.e. $\text{NN}_{\text{output}}(p) = \text{dist}(p, g)$

**Case 1** $p$ added after $g$ then $\text{NN}_{+}(p) = \text{NN}_{\text{output}}(p)$

$$d_{fs}(p) \leq c_{e} \text{NN}_{+}(p) = c_{e} \text{NN}_{\text{output}}(p)$$

**Case 2** $g$ added after $p$.

$$\text{NN}_{+}(g) \leq \text{dist}(p, g)$$

$$d_{fs}(p) \leq d_{fs}(g) + \text{dist}(p, g) \leq c_{e} \text{NN}_{+}(g) + \text{dist}(p, g)$$

$$\leq (c_{e+1}) \text{dist}(p, g)$$

$$= (c_{e+1}) \text{NN}_{\text{output}}(p)$$
Thm. DR generates a mesh with at most
\[ C \int_{B_{\pi}} \frac{1}{d(x)} \, dA \] vertices.

\[ \frac{d(x)}{d(x) + r_p} \]

Note. Balls \( B(P, r_p) \) are disjoint.

Note. Max \( d(x) \leq d(x) + r_p \)

\[ \int_{B_{\pi}} \frac{1}{d(x) + r_p} \, dA \geq \text{Area}(B_p) \left( \frac{1}{(d(x) + r_p)^2} \right) = \frac{\pi r_p^2}{(d(x) + r_p)^2} = \frac{\pi}{(2n+3)^2} = C' \]
\[ \int_{\text{Box}} \frac{1}{\Delta s^2(x)} \, d\mathcal{A} = \sum_{p \in \mathcal{V}(D)} \frac{1}{4} \int_{B_p} \frac{1}{\Delta s^2(x)} \, d\mathcal{A} \]

\[ \geq \sum_{p \in \mathcal{V}} \frac{1}{4} c' = \frac{1}{4} c' |\mathcal{V}| \]