CG

10/25/12

Delaunay Refinement

Ruppert

2D only

<u>Input</u>: PSLG all angles $> 60^\circ$ (G)

<u>Output</u>: 1) 2D simplicial complex

2) A refinement of the PSLG

3) Delaunay

4) no small angle

5) constant times opt size

<u>Def</u> lfs(x) = dist to 2nd nearest disjoint feature.

<u>Def</u> $f: R^d \to R$ is α-Lipschitz if $\forall p, q \in R^d$

$$|f(p) - f(q)| \le \alpha \, \mathrm{dist}(p, q)$$

<u>Claim</u> lfs is 1-Lipschitz i.e.

$$\mathrm{lfs}(p) \le \mathrm{lfs}(q) + \mathrm{dist}(p, q)$$

Algorithm

Def Circumball of a simplex is min radius ball B with vertices on ∂B.

Def p encroaches simplex S if $p \in \text{Inter}(\text{Circumball}(S))$
S' encroaches on S if $\text{circumcenter}(S') \in \text{Inter}(\text{Circumball } S)$

Def segment is a subsegment of an edge of G.

Delaunay Refinement (G) (Overview)

1) Add a bounding box to G
2) Compute Delaunay of $V(G) + \text{Box}$
3) While
   1) a segment is encroached add circum-center
   2) a Δ is skinny add circum-center.

Algorithm Details

Subroutine: Split(seg s, S)

1) Add circumcenter of s to V & update DT(V)

2) a) remove s from S

b) add halves of s to S.

<u>Delaunay Refinement</u> (PSLG G, angle α or radius-edge ρ)

<u>Init</u> 1) Add bding box to G

2) $S = \text{edges}(G)$

3) $V = \text{vertices}(G)$

4) $T = DT(V)$

While $\exists$ encroached seg or skinny tri do

1) While $\exists s \in S$ (s encroached) splitseg(s)

2) <u>If</u> t skinny (radius-edge $> \rho$) <u>then</u> <u>do</u> (*)

<u>If</u> t encroaches a seg s <u>then</u> <u>do</u>

splitseg(s)

<u>else</u> splittri(t)

Return $DT(V)$

<u>Def</u> $NN_t(P) \equiv$ nearest vertex in V at last time P was considered for insertion before t.

e.g. at step (*) $P \equiv$ circumcenter(t) was considered but may not have been added.

<u>Def</u> <u>Containing Dimension</u> of $P \equiv$ min dim feature containing P.

eg P is an input point then $CD(P) = 0$

P <u>interior</u> to an edge $CD(P) = 1$

o.w. $CD(P) = 2$

<u>Lemma</u> $\exists$ constants C_e & C_t depending only on α s.t. $\forall t$

1) <u>If</u> $CD(P) = 0$ <u>then</u> $lfs(P) \leq NN_t(P)$

2) If $CD(P) = 1$ then $lfs(P) \leq C_e NN_t(P)$

3) If $CD(P) = 2$ then $lfs(P) \leq C_t NN_t(P)$

<u>pf</u> Induction on execution time, t.
Assume t and show t+1

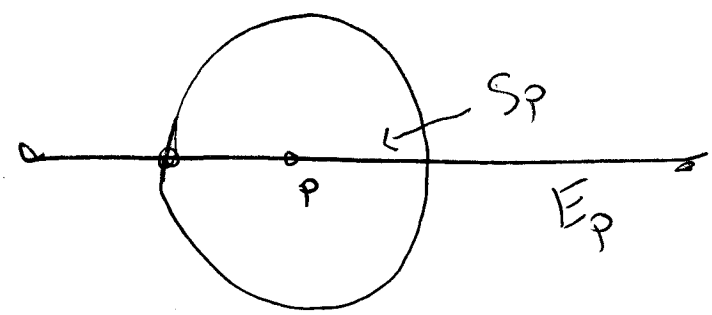
<u>Case</u> $CD(p)=0$ (p will only be considered once)

$$lfs_G(p) \le lfs_V(p) \le NN_t(p) = NN_{t+1}(p)$$

Case $CD(p)=1$

Let $p \in E_p$ (input edge) & p = circumcenter of segment S_p

ie $p \in S_p \subseteq E_p$



S_p must have been encroached by some point a

Pick $a \in Ball(S_p)$ as follows.

1) $\exists a \in Ball(S_p)$ at time splitseg(S_p)
set a to closest such point to p.

<u>else</u> let $a \in B(S_p)$ be circumcenter of skinny tri that yielded to p.

<u>Subcase</u> $CD(a) = 0$

$$lfs(p) \underset{CD(a)=0}{\leq} dist(p, a) \leq NN_{t+1}(p)$$

a was closest point.

we need $\boxed{1 \leq C_e}$

<u>Subcase</u> $CD(a) = 1$

Let $a \in E_a$ (input edge)

(Lets assume input angles $\geq 90^\circ$)

Thus $E_a \cap E_p = \emptyset$

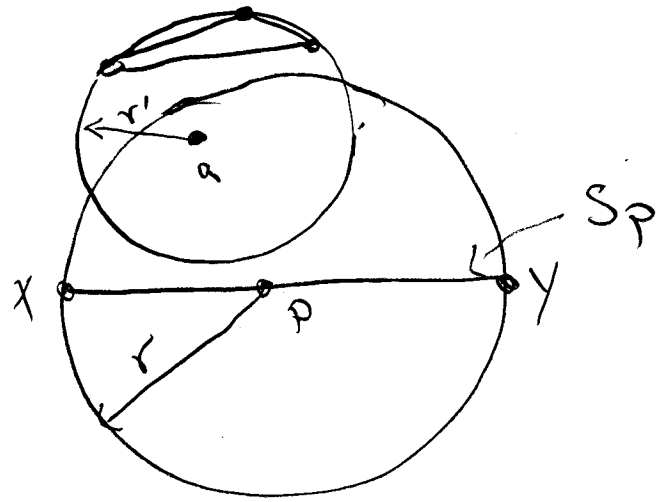
$$lfs(p) \leq dist(p, E_a) \leq dist(p, a) = NN_{t+1}(p)$$

$$\boxed{1 \leq C_e}$$

(60° case?)

<u>Subcase</u> $CD(a) = 2$

By induction $lfs(a) \le C_t \, NN_t(a) \le C_t r'$



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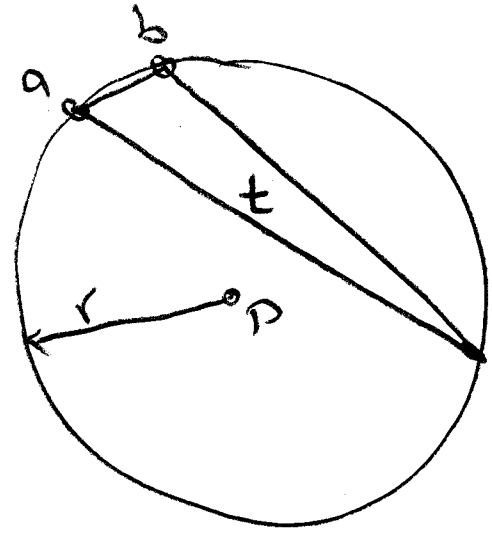
1) $NN_t(p) = r$ (a will yeild to S_p)

2) $a \in B(p, r')$ & $x, y \notin B(a, r') \Rightarrow r' \le \sqrt{2}\, r$

$$lfs(p) \le lfs(a) + dist(p, a) \le C_t r' + r$$
$$\le C_t \sqrt{2}\, r + r$$
$$\le (\sqrt{2}\, C_t + 1)\, NN_{t+1}(p)$$

We need $\boxed{1 + \sqrt{2}\, C_t \le C_e}$



<u>Case</u> $CD(p)=2$

<u>WLOG</u> a added before b.

<u>Subcase</u> $CD(b)=0$

$lfs(p) = r = NN_{t+1}(p)$ $\boxed{1 \le C_t}$

<u>Subcase</u> $CD(b)=1$

(induct) $lfs(b) \le C_e NN_t(b) \le C_e \, dist(b,a)$

radius-edge $\rho \le r/e$ ie $e \le r/\rho$ $\bar{\rho} = 1/\rho$

$e \le \bar{\rho} r$

$$
\begin{aligned}
lfs(p) \le lfs(b) + dist(p,b) &= lfs(b) + r \\
&\le C_e \, dist(b,a) + r \\
&< C_e \bar{\rho} r + r = (C_e \bar{\rho} + 1) r \\
&= (C_e \bar{\rho} + 1) NN_{t+1}(p)
\end{aligned}
$$

need $\boxed{(C_e \bar{\rho} + 1) \le C_t}$

<u>Subcase</u> $CD(b)=2$

Same as last case but C_e is C_t

ie $(C_t\bar{\rho}+1) \leq C_t$

$$\frac{1}{1-\bar{\rho}} \leq C_t \quad \text{or} \quad \frac{\rho}{\rho-1} \leq C_t$$

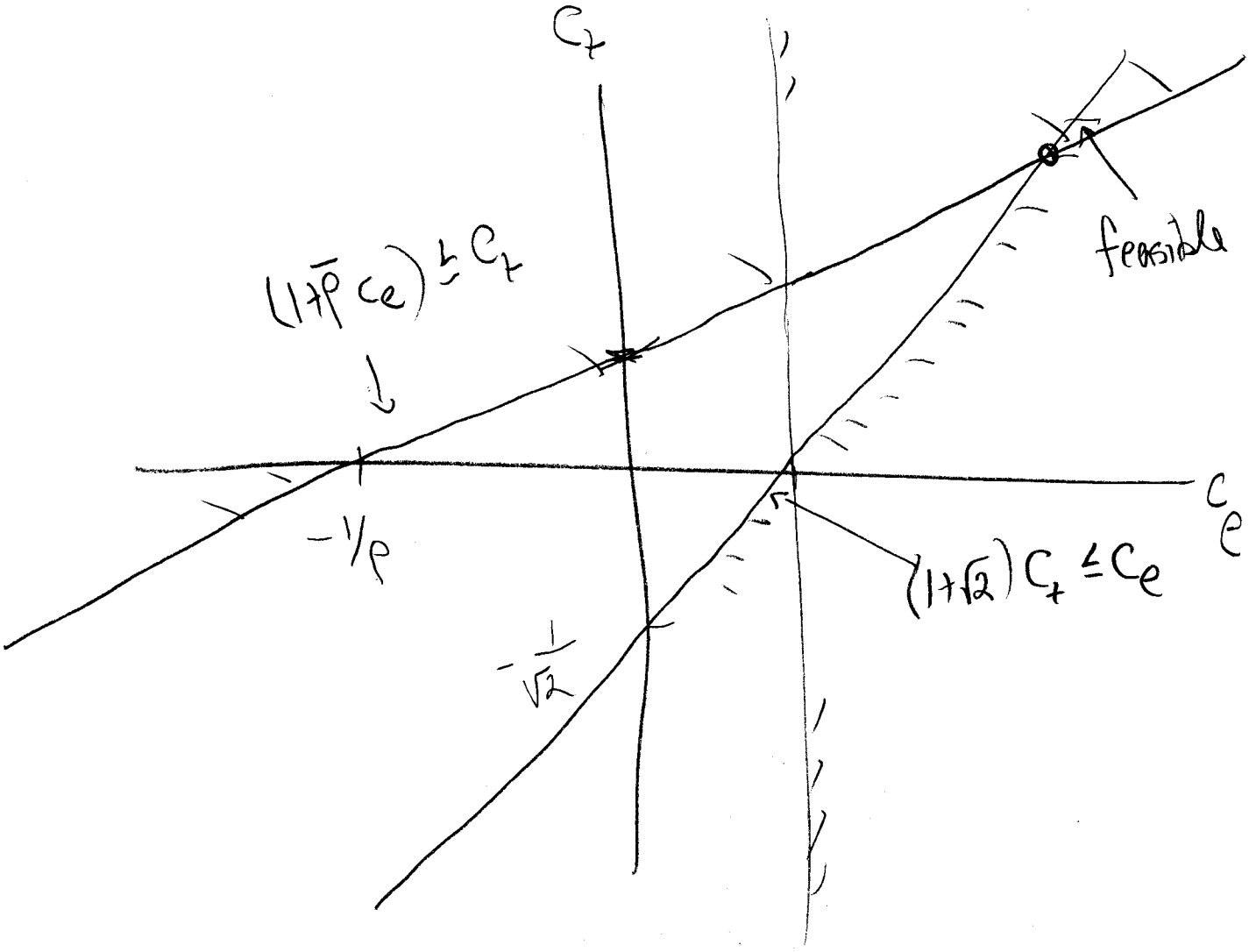
Our list of needed conditions for ρ

$1 \leq C_e$

$\frac{\rho}{\rho-1} \leq C_t$

$1+\sqrt{2}\,C_t \leq C_e$

$1+\bar{\rho}\,C_e \leq C_t$



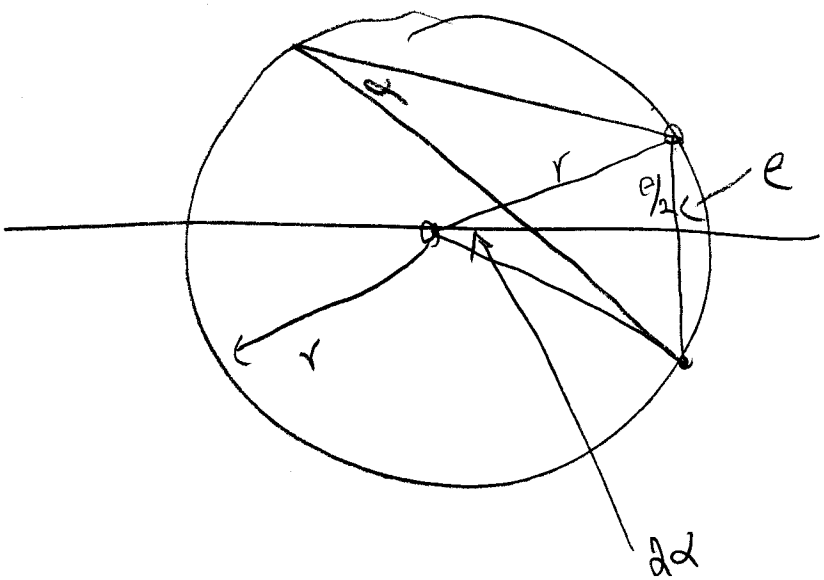
$C_t = \frac{C_e}{\sqrt{2}} - \frac{1}{\sqrt{2}}$ slope $\frac{1}{\sqrt{2}}$

$C_t = \bar{\rho}\, C_e + 1$ slope $\bar{\rho}$

$\Rightarrow \bar{\rho} \leq \frac{1}{\sqrt{2}}$

or $\rho \geq \sqrt{2}$

$$\rho = r/e \geq \sqrt{2}$$



$$\frac{e/2}{r} = \sin\alpha$$

$$\left(\frac{1}{2\sqrt{2}}\right) = \sin\alpha \quad \Rightarrow \quad \alpha \approx \sin^{-1}\left(\frac{1}{2\sqrt{2}}\right) \approx 20^0$$

<u>Thm</u> $p \in$ Output then $lfs(p) \leq (c_e+1) NN_{output}(p)$

<u>pf</u>

Let q be NN of p in output ie $NN_{output}(p) = dist(p,q)$

<u>Case 1</u> p added after q then $NN_t(p) = NN_{output}(p)$

$$lfs(p) \leq c_e NN_t(p) = c_e NN_{output}(p)$$

<u>Case 2</u> q added after p.

$$NN_t(q) \leq dist(p,q)$$

$$lfs(p) \leq lfs(q) + dist(p,q) \leq c_e NN_t(q) + dist(p,q)$$

$$\leq (c_e+1)\, dist(p,q)$$

$$= (c_e+1)\, NN_{output}(p)$$

<u>Thm</u> DR generates a mesh with at most

$$C \int_{B_{1x}} \frac{1}{lfs^2(x)}\, dA \text{ vertices.}$$

<u>pf</u> Let $r_p = \dfrac{lfs(p)}{2(c_e+1)}$

<u>Note</u> Balls $\underset{\underset{B_p}{\shortparallel}}{B(p, r_p)}$ are disjoint.

<u>Note</u> $\displaystyle\max_{x \in B_p} lfs(x) \le lfs(p) + r_p$

$$\int_{B_p} \frac{1}{lfs^2(x)}\, dA \ge \text{Area}(B_p) \frac{1}{(lfs(p)+r_p)^2} = \frac{\pi r_p^2}{(2(c_e+1)r_p + r_p)^2}$$

$$= \frac{\pi}{(2c_e+3)^2} \equiv C'$$

$$\int_{\text{Box}} \frac{1}{\mathrm{lfs}^2(x)}\, dA \geq \sum_{p \in V(D)} \frac{1}{4} \int_{B_p} \frac{1}{\mathrm{lfs}^2(x)}\, dA$$

$$\geq \sum_{p \in V} \frac{1}{4} C' = \frac{1}{4} C' |V|$$