Chernoff Bounds for Traps

Let $X_1, \ldots, X_t$ be independent 0/1 random variables.

Assume $\Pr(X_i = 1) = p$.

The binomial random variable is $S_n^p = X_1 + \cdots + X_t$.

$$\Pr(S_n^p \in (1-\beta)p^t, \beta p^t)$$

$$\text{Expect}(S_t^p) = \sum E(X_i) = p \cdot t$$
Theorem 2

\[ \text{Prob}\left( S_t^p < (1-\beta)p_t \right) < e^{-\beta^2 p_t/2}, \quad \forall 0 \leq \beta < 1 \]

\[ \text{Prob}\left( S_t^p > (1+\beta)p_t \right) < e^{-\beta^2 p_t/2}, \quad \forall 0 \leq \beta < 2 \]
**Theorem**

If $N$ non-crossing bars separate queries, query $q$, $\lambda > 0$

\[ \text{Prob}(\text{search}(q) > 3\lambda \ln(n+1)) \leq \frac{1}{(n+1)^{\lambda \ln(1.25)-1}} \]

**Proof**

Let $Z$ be random variable # nodes on search path.

**Goal:** Write $Z$ as sum of independent variables.

Consider DAG of all subsets of $\{1, 2, \ldots, N\}$

![Diagram of a DAG with subsets](attachment:diagram.png)
Each path in an insertion order!

Mark edge if trap contains 9 changes.

Indegree \leq 4 by BA

Mark more edges for \( i \geq 4 \) so indeg is 4.

**Def** \( X_i = \begin{cases} 1 & \text{if } \text{ith edge is marked} \\ 0 & \text{otherwise} \end{cases} \)

\( z \leq 3y \) where \( Y = \sum X_i \)

**Goal's bd** \( \Pr \left[ Y \geq 1.1 n \ln(n+1) \right] \)

**Thm (Markov)** \( \Pr \left[ X \geq x \right] = \frac{E[X]}{x} \quad x \geq 0 \)
\[ E(Y) = \sum E(X_i) = 4H_n \]

\[ E(X_i) \leq \frac{4}{i} \quad (i \text{ small ?}) \]

Note: \( X_i \) are i.i.d.

\[ \text{Chernoff: } \quad \text{Prob} \left( Y > (1+\beta)E(Y) \right) \leq e^{-\beta^2 \frac{4}{3} H_n} \]

\[ = (e^{H_n})^{-\frac{4}{3} \beta^2} \quad \text{Wolfram: } \quad e^{H_n} \leq n+1 \]

\[ \leq \frac{1}{(n+1)^{\frac{4}{3} \beta^2}} \quad (\text{Concentration}) \]

Claim: At most \( 2/(n+1)^2 \) different \( g \)s.

Proof: \( 2(n+1) \) slabs using \( 2(n+1) \) endpoints

Each slab has \( (n+1) \) traps

Note: \( 2g \) in each trap have same search!
Prob that some search is more than

\[(1+\beta)^4 A_n \in \frac{1}{(n+1)^{\frac{3}{2}\beta - 2}}\]

Pick \( \beta \) s.t. \( \frac{3}{2}\beta - 2 \geq 1 \)

\[\frac{3}{2}\beta \geq 3\] or \( \beta \geq \frac{3}{2} \)

\[\beta^* = \frac{3}{2}\]