The Convex Hull Prob
(Sorting Prob of CG)

Def A \in \mathbb{R}^d \text{ convex if closed under convex combinations.}

Def ConvexClosure(A) = \text{CC}(A) = \text{smallest convex set} \supseteq A

2 Defs of Convex Hull

Def 1 \text{ } \text{CH}(A) = \text{CC}(A)

Def 2 \text{ } \text{CH}(A) = \text{CC}(A)

We will use Def 1

A finite set

Thus in 2D \text{CH}(A) \text{ is a simple closed polygon. (say CCW)}

\begin{tikzpicture}
  \draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\end{tikzpicture}
We will use following characterization

Claim \( [a, b] \) is on \( \text{CH}(A) \) iff \( a \neq b \)

1) \( a, b \in A \)
2) \( \forall a' \in A \) either \( a' \) left of \( [a, b] \)
   or \( a' \in [a, b] \)

2D Convex Hull by divide-and-conquer

\( A = \{P_1, \ldots, P_n\} \quad P_i = (x_i, y_i) \)

Preprocess: sort \( A \) by \( x \)-coordinate

2D-CH(\( A \))

\( \text{if} \ |A| = 1 \quad \text{return} \ P_i \)

\( \text{else} \quad \text{CH}_L = 2\text{D-CH}(P_1, \ldots, P_{n/2}) \)
\( \text{CH}_R = 2\text{D-CH}(P_{n/2+1}, \ldots, P_n) \)

STITCH(\( \text{CH}_L, \text{CH}_R \))
STITCH \((L, R)\)

Lower bridge \((L, R)\)
- \(a = \text{right most } L\)
- \(b = \text{left most } R\)

Repeat **)**:**

* While \(a \text{ Right}(a, b)\) set \(a \leftarrow a\)

** while \(b \text{ Right}(a, b)\) set \(b \leftarrow b\)

Upper bridge \((L, R)\) = ?

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**Correctness**

* generates triangles \((a, b, a)\)

**"**" \((a, b, a)\)

1. The \(\Delta\)'s are disjoint
   - They are ordered by their intersection with vertical line \(b\).
2. They are in \(CC(A)\).

Thus termination:

At termination \(a, \bar{a}, b, \bar{b}\) are all lift of \((a, b)\)
Since \((a, a), (a, b), (b, b), (b, b)\) are on \(CH(L)\) & \(CH(R)\) respectively.

Done

\[\text{Timing: Preprocess } O(n \log n) \text{ to sort} \]

\[\text{STITCH in } O(n) \]

\[T(n) = 2T(n/2) + c \cdot n \]

\[T(n) = O(n \log n) \]
Lower bounds

Sorting reducible to $\text{CH}$

Input: $x_1, \ldots, x_n$

$\text{CH}(x_1, x_2, \ldots, x_n, x_2)$

The CH will be $x_i$'s in sorted order.

An important use for $\text{CH}$

$p_i, \ldots, p_n \in \mathbb{R}^2$

$\tilde{p}_i = (p_x, p_y, p_x^2 + p_y^2)$

$\text{CH}(\tilde{p}_1, \ldots, \tilde{p}_n) = \text{Triangulated surface}$

The Delaunay Triangulation
Random Incremental CH

Procedure Random Incremental CH(P)

0) Make $\Delta = \langle P_1, P_2, P_3 \rangle$ pick $C$ in interior $\Delta$

1) Construct ray from $C$ to each $P_i$

2) Partition $P_i$ by edge of $\Delta$ they cross.

3) Randomly permute $P_1, \ldots, P_n$.

For $i=1$ to $n$

let $e$ be edge crossed by ray $C \rightarrow P_i$

BuildText($P, e$)

Procedure BuildText($P, e$)

1) Find edges of CH "visible" to $P$ by searching out from $e$.

a) Replace visible edges with 2 new edges.

3) Assign rays to the new edges.
An Example

$n$-points on a circle

Worst case: Incremental order $P_1 \rightarrow P_n$

"Best" case $P_1, P_2, P_3, P_{n/2}, P_n, P_{3n/4}, \ldots$
Correctness?

Timing

$O(n)$ work other than BuildTest.

Consider steps 1 & 2 in BuildTest

1) At most an edge generated over life of algo.
   a) Charging rule for line-side tests
      i) not visible tests: we charge $P_i$
         each visible test: we charge to the edge
      total $2n$ + $2n_i$ on $4n$ tests.

Consider step 3 in BuildTest:

Ray-costs

Backwards analysis

i-3 points to pick from say $P_i$

$\text{Cost}(P_i) = \begin{cases} 0 & \text{if } P_i \text{ not on hull} \\ \# \text{ray crossing to left} \& \text{right} & \text{otherwise} \end{cases}$
\[ \mathcal{C}_i = \text{cost} \]

\[ E(\mathcal{C}_i) \leq \frac{2(n-i)}{i-3} \]

\[ \mathcal{C} = \text{total cost} \]

\[ \mathbb{E}(\mathcal{C}) = \sum_{i=1}^{n} \mathbb{E}(\mathcal{C}_i) \leq \sum_{i=1}^{n} \frac{2(n-i)}{i-3} \leq 2n \sum_{i=1}^{n} \frac{1}{i} \]