Computing 2D Delaunay

CG
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Lower Bound $\Omega(n \log n)$

Many Optimal algorithms

1) One approach

   a) Compute the Voronoi using sweep line
      $O(n \log n)$ BKOS Chap 7

   b) Dualize obtaining Delaunay $O(n)$

2) Divide-and-conquer

3) Random Incremental BKOS Chap 9

Random Incremental Delaunay

Input: $P \subseteq R^2$ $|P| = n$

Big Triangle P_{-1}, P_{-2}, P_3 containing P

Simple Delaunay Tri (P)

1) Make big Δ containing P
2) Init: T to Δ and assign P to Δ
3) Randomly order $P = \{P_1, \dots, P_n\}$
4) For $i=1$ to n
   a) Find tri $t \in T$ containing P_i
   b) (Form cavity) Remove all tri $t' \in T$ s.t. $P_i \in C(t')$
   c) (Form tent) $\forall e = (a,b)$ on ∂ Cavity form tri (P_i, a, b)

Delaunay Tri (P)

Replace 4) with 4')

4') For $i = 1$ to n

a) Find $t \in T$ containing P_i

i) If P_i is interior to t then split t into 3 Tri.

ii) If P_i is on an edge between t & t'

split into 4 tri

b) While $\exists$ an edge opposite P_i is illegal.

1) Flip the edge

2) Update data struture

Claim Delaunay Tri is correct

i.e. If step 4' starts with a Delaunay, it returns a Delaunay.

The only illegal edges must "see" P_i

All new edges are Delaunay.

Timing

Lemma Expected # Tri created is at most $9n+1$

$V_r \equiv$ # tri at step r

Backwards Analysis

if d = degree of removed vertex

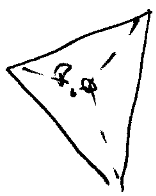
$V_r = 2d-3$ Note Euler $E(d) < 6$

$E(V_r) < 2 \cdot 6 - 3 = 9$

Thm Delaunay Tri is $O(n \log n)$ expected time & $O(n)$ expected storage.

Storage $\equiv$ # created tri $= O(n)$

Cost of point location?

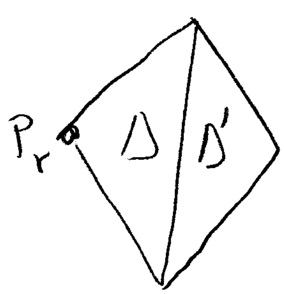


2 line side tests per point in triangle

Flip $\equiv$ 1 lineside test per point in both triangles

Def $C(\Delta) \equiv$ circum circle of Δ

Note Each flip starts with a new tri Δ & an old Δ'.



$\therefore\; q \in \Delta \cup \Delta' \Rightarrow q \in C(\Delta')$

Def $K(\Delta) = \{ q : q \text{ interior } C(\Delta) \}$

lineside tests $= O\left(\sum_{\Delta} |K(\Delta)|\right)$

order cost by time tri was created.

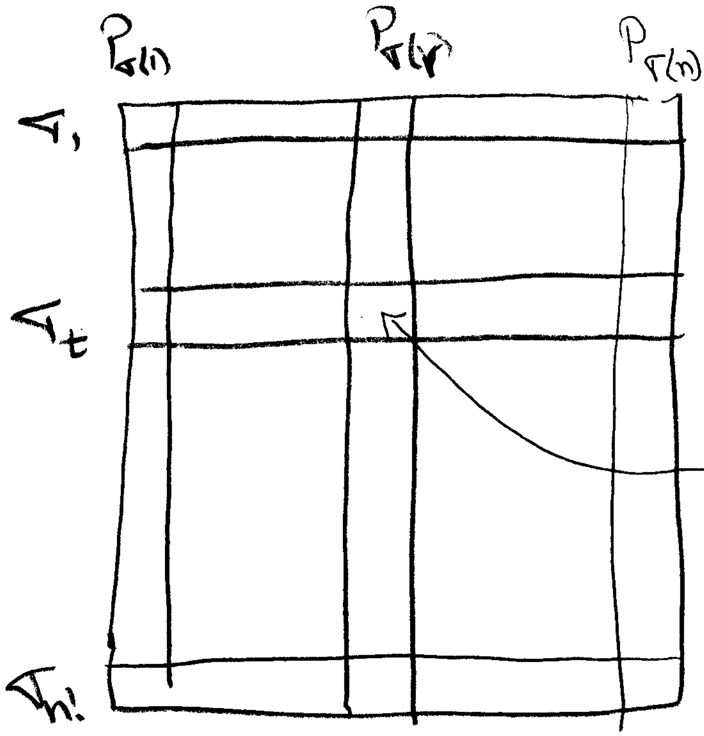
Let $P_r \subseteq P \wedge |P_r| = r \wedge \mathcal{T}_r = \mathrm{Del}(P_r)$

$$\#\text{ lineside tests} = \sum_{r=1}^{n} \sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} |K(\Delta)|$$

<u>Def</u> $K(P_r, q) = |\{\Delta \in \mathcal{T}_r : q \in C(\Delta)\}|$

$K(P_r, q, p)$ " " $\&\ p$ vertex $\Delta\}|$

Let view cost as a table



$\sigma_i \in S_n$

*

fix σ

$\sigma = \sigma_t \quad P'_i = P_{\sigma(i)}$

$\bar{P}'_r = \{P'_1, \dots, P'_r\}$

Cost $= \sum_{j>r} K(\bar{P}'_r, P'_j, P'_r)$

Goal: Expected cost of a row

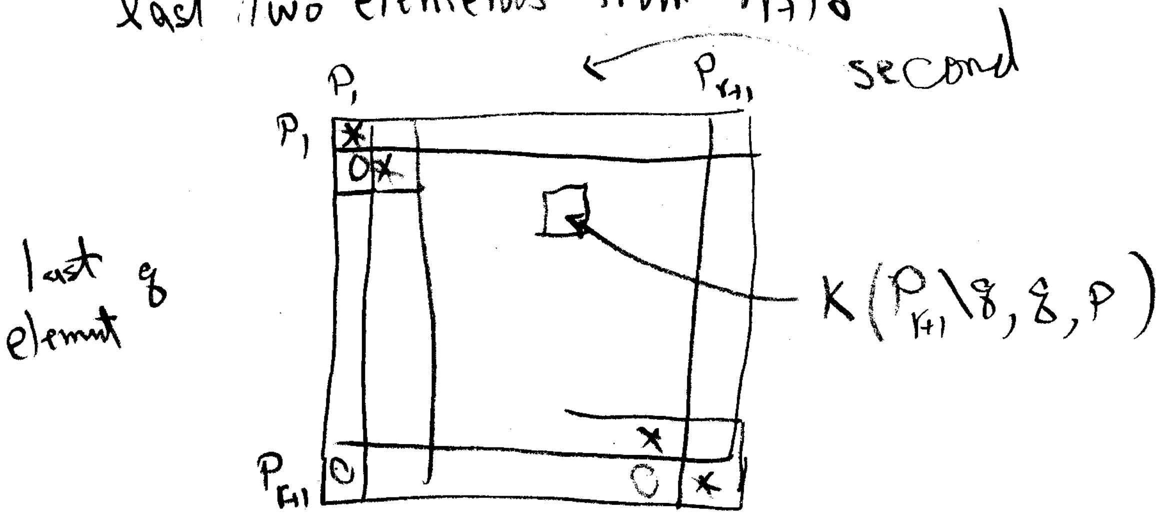
Instead: Expected cost of rth column & sum

<u>Def</u> Equivalence relation on S_n

$\sigma \equiv_r \sigma'$ if $\{\sigma(1), \dots, \sigma(r)\} = \{\sigma'(1), \dots, \sigma'(r)\}$

We will uniformly bound each equivalence class.

Suffice to fix a set P_{r+1} and pick the last two elements from P_{r+1}.



<u>Note</u> Row q sums to $3K(P_{r+1}\setminus q, q)$

<u>Def</u> $\deg(q)$ = degree of q in $\mathcal{T}_{r+1}$

<u>Thus</u> $K(P_{r+1}\setminus q, q) = \deg(q) - 2$

$$\therefore \sum_q K(P_{r+1}\setminus q, q) \leq \sum_q (\deg(q) - 2) =$$

$$= \sum_q \deg(q) - 2(r+1) < 6(r+1) - 2(r+1) = 4(r+1)$$

Sum over the Table

$$\sum_{p,q} K(P_{r+1} \setminus q, q, p) < 3(4(r+1)) = 12(r+1)$$

$$\text{Expect}\left(K(P_{r+1} \setminus q, q, p)\right) \le \frac{12(r+1)}{(r+1)r} = \frac{12}{r}$$

Back to the first table

Expected value of column $r < \dfrac{12(n-r)}{r}$

$$\text{Expected row} < \sum_{r=1}^{n} \frac{12(n-r)}{r} = 12nH_n - 12n$$

$$= 12n(H_n - 1)$$