Graph Rigidity

Def. $G$ is rigid in $\mathbb{R}^2$ if some embedding is infinitesimally rigid.

Laman: $G$ is minimally rigid iff

1) $m = 2n - 3$
2) $\forall H \subseteq G$ $m' \leq 2n' - 3$

$K_{3,3}$

$\text{Prim}$

not inf-rigid

inf-rigid
Let $G$ be infinitely rigid if

$$\forall v = (v_1, \ldots, v_n) \in \mathbb{R}^n \text{ if } (v_i - v_j) \cdot (p_i - p_j) = 0 \quad \forall (i, j) \in E$$

then $V \equiv 0$

Let $G$ be minimally rigid if

$G$ is inf-rigid & $G \not\in \text{ not } \text{Vec} E$. 
Lovász-Yemini 82: \( G \) is minimally rigid iff \( G + \text{edge} \cong 2 \text{ edge disjoint spanning trees.} \)
Henneberg 1860: $G$ is min rigid iff
$G$ can be obtained using Henneberg's rules

Rule 0: Start with an edge.
Rule 1: Add vertex of degree 2

Rule 2: 1) Remove $(v,w)$
2) Add $x$ adjacent to $v,w$, if $e \not\in V$

Diagram:

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es

[Diagram of a triangle with vertices connected to form a larger triangle, then to a simpler structure]
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Claim: If the embedded graph \( m < 2n - 3 \) then \( T \) is not rigid.

\[ V = \{ v_1, \ldots, v_n \} \] is not a rigid motion.

\[ (v_i - v_j) \cdot (p_i - p_j) = 0 \quad \forall (i, j) \in E \]

\[ \begin{pmatrix}
    V_1^x \\
    V_1^y \\
    \vdots \\
    V_n^x \\
    V_n^y
\end{pmatrix} = \begin{pmatrix}
    0 \\
    0 \\
    \vdots \\
    0 \\
    0
\end{pmatrix} \]

\( m + 3 < 2n \) done!
Lemma 6. If $G$ is Laman then $G$ has a Henneberg construction.

By Induction on $n$. $n > 3$

$m = 2n - 3 \quad \forall v \in V$ of degree $\leq 3$

If $\deg(v) = 2$ remove it

Then $G \setminus v$ is Laman

If $\deg(v) = 3$ then $G \setminus v \quad m' = 2n' - 4$

then $G'$ is not rigid

Compute maximal rigid components of $G'$

$G'_1, \ldots, G'_x$

$v$ must attach to at least 2 $G'_i$

else $G' \cup \{v\}$ has too many edges.
Pseudo-Tri & Rigidity

Pseudo-tri examples

Pseudo-Δ

Vertex types
convex - corners
straight
reflex

inside is P-3-gon
outside is P-4-gon

Pseudo-triangulation (including CH) P-Δ

Triangulating a Polygon
Suppose $G$ is embedded planar.

Def: A vertex is **pointed** if some angle at $x$ is reflex.

$G$ is **pointed** if all $v_e V$ are pointed.
Lemma: If $T$ is a $P\Delta$ of $n$ points $P$ then

$$m = |E(T)| = 2n - 3$$

$$t = \#\Delta\cap T \quad m = \#\text{edge}$$

Euler: $t - m + n = 1$ (no outside face)

$$\#\text{angles} = 2m$$

$$\#\text{corners} = 2m - n \quad (\text{pointed condition})$$

$$3t = 2m - n$$

Euler: $3t - 3m + 3n = 3$

$$2m - n - 3m + 3n = 3$$

$$-m + 2n = 3$$

$$m = 2n - 3$$
Theorem: Since pointed PD of a CH theorem
is rigid.

We consider dual problem!

Maxwell lifting

Claim: If $f$ is a lifting of a face $F$ & $P$ is a reflex vertex of $F$ then

$P \in \operatorname{Max}(F)$ iff $f$ on $F$ is constant.
Thm If $G$ is a pointed $\mathcal{P}_\Delta$ & $e \in \text{CH}(G)$ then expanding $e$ gives an expansive motion of $G$. 
Converting a Polygon

Alg (R polygon)

1) Add struts to & forming pointed PD (convex)
2) While 3 struts s on CH do
   a) Expand s until some pair e & e' are straight
   b) If e & e' Bars then freeze e & e'
      else clean up by edge flipping