Mesh Generation

Goal: Partition domain into simplices.

Simplex: 1

Segment, triangle, tetrahedron

Partition: Intersection of 2 simplices in a simplex

Conforming: to input

Well-shaped simplices

a) no small angles $\leq 0^\circ$

b) no large angles $\geq 80^\circ$

Small number of simplices (optimal size)
Repeat Ratio

\[ A(a, b, c) = \frac{\text{longest-side}}{\text{alt}} \]

\[ R(a, b, c) = \frac{\text{longest-side}}{\text{shortest-side}} \]

\[ \frac{1}{\text{Smallest-angle}} \]

\[ \frac{1}{180^\circ - \text{largest-angle}} \]

radius-edge ratio = \( r/e \)

\( r = \text{radius of circum sphere} \)

\( e = \text{shortest edge} \)
Mesh Generation Methods

1) Quadtree (today)
2) Delaunay Refinement (Thursday)
3) Advancing Front
4) Ball-Packing

In 2D our input will be PSLG.
Simplex & Simplicial Complex

Def: \( P_0, \ldots, P_k \in \mathbb{R}^d \) are affinely independent of dimension \( k \) if \( P_i - P_0, \ldots, P_k - P_0 \) are independent.

Def: If \( P_0, \ldots, P_k \) are \( a \)-ind then \( CC(P_0, \ldots, P_k) \) is a \( k \)-simplex & \( \forall S \subseteq \{P_0, \ldots, P_k\} \) \( CC(S) \) is a sub-simplex.

Def: A set \( K \) of simplices in \( \mathbb{R}^d \) is a Simplicial Complex if:

1) \( K \) is closed under sub-simplex

2) \( S \subseteq K \) then \( S \subseteq \) sub-simplex of \( K \)

Def: \( \text{dim} \,(K) = \max \text{dim} \, S \in K \)
Note: $PSLC$ is a 1-dim simplicial complex in $\mathbb{R}^2$

$K$ & $K'$ are simplicial complexes

**Def**: $K'$ is a *refinement* of $K$ if

$$\forall s \in K \text{ of dim } k \exists s_1, \ldots, s_r \in K' \text{ of dim } k$$

$$\text{st. } S = \bigcup_{i=1}^{r} s_i$$

Input: Simplicial complex $K$ & Domain $\mathcal{N}$

$$\text{se} K = \{ s \in \mathcal{N} \}$$

Output: refinement $K'$ of $K$ s.t.

$$\bigcup_{\text{se} K'} U S = \mathcal{N}$$
Quad-Tree Meshing

Input: set $X \subseteq \mathbb{R}^2$ of points $X \subseteq B$ (box) $|X| = n$

Def: QT is a tree of nested square boxes. The children of box $b$ are either
1) empty (leaf box)
2) 4 children of half the size (split of $b$)

Neighbors: 4 direct neighbors
8 extended neighbors
**Definition**

A QT is balanced if every leaf box has no side containing more than one interior node.

![Diagram showing an OK example](image1)

![Diagram showing a not OK example](image2)
Build QT (X, B)

Init: QT T = (X, B)

1) While 3 leaf box (x', b) at bin crowded
   split b and assign x' to new boxes.

2) Balance T by splitting.

3) Split all boxes containing a point
   has 8 extended neighbors (leaf boxes).
Def: Let box $b$ be crowded if $\exists x \in b$ and one of the following holds:

C1) $\exists y \neq x \in b$
C2) $\exists y \in X$ s.t. $\text{dist}(x,y) \leq 2\sqrt{2} \cdot \text{side length}(b)$
C3) an extended neighbor of $b$ is split.

\[ \]

Warping

$x \in b$ to closest corner of $b$ then warp $y$.

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\]

$b$ empty & not warped

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Thm: \( T \) is a QT & \( T' \) in its balanced version 
then \( \| T' \| = O(\| T \|) \) \( \| T' \| = \# \) boxes

PS

note: \( T \) is a proper \( k \)-ary tree

Note: A proper \( k \)-ary tree with \( i \) internal nodes has size \( k \cdot i + 1 \) (induct)

boxes of \( T \) are old.

Claim: A new internal box has an extended neigh which is old.

Proof: by contraction
Let \( b \) be the smallest internal new box with no old neighbors.

\[ b_{\text{internal}} \Rightarrow \text{a side of } b \text{ is split twice} \]

**Example**

\[  \]

\[ b' \text{ is new with no old neighbors. Contra!} \]

\[ \#	ext{int}(T') \leq 8 \cdot n \quad \text{(?)} \]

\[ \#	ext{(T')} \leq 4 \cdot \#	ext{(int(T')} + 1) \leq 32n + 1 \]

**Thm** Balanced QT can be computed in \( O(dn) \)

Time \( d_1 = \text{depth} \)