

Ball Packings

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Don Sheehy
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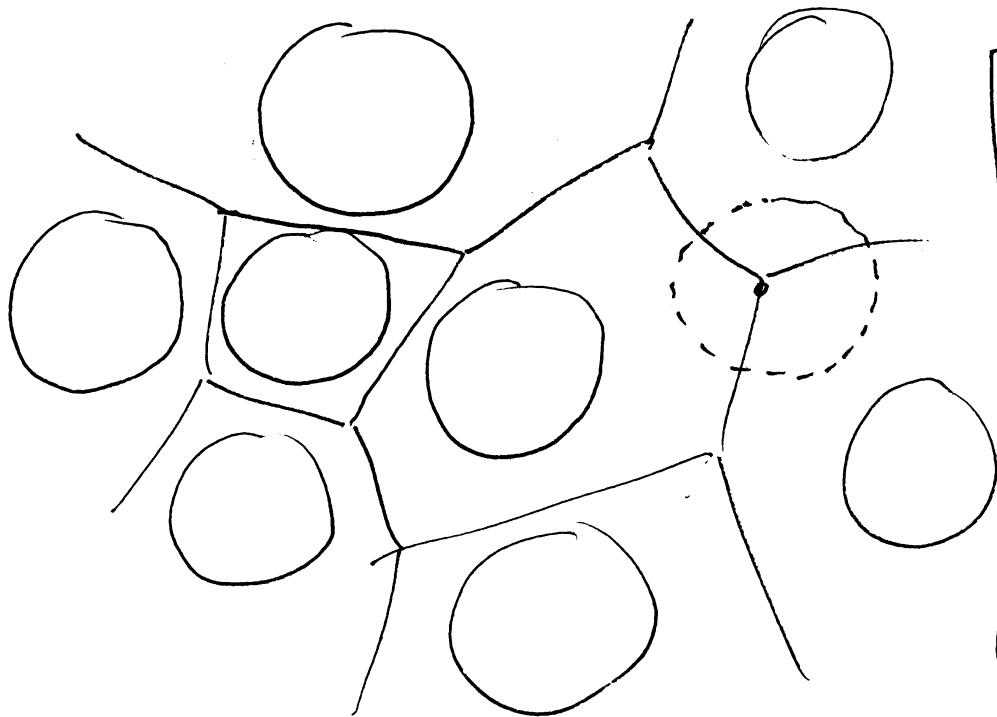
Def: Packing \equiv interiors are pairwise disjoint.

Def: Maximal \equiv no more will fit.

Goal: Find a Small maximal packing.

Greedy Algorithm: - Find some empty space \leftarrow How!?
- Put a ball there.

Greedier Algorithm - Find the biggest empty space
- Put a ball there.



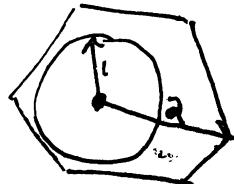
Key Idea:
Voronoi Diagram
encodes the
big open spaces.

Problem: Given n points in a $k \times k$ box, how many total will there be after running the greedier algorithm?

Upper Bound Total volume of the balls is at most k^2 .

$$\Rightarrow m \leq \frac{k^2}{\pi} \quad \begin{matrix} \nearrow \\ \text{output size} \end{matrix} \quad \begin{matrix} \nwarrow \\ \text{volume of a ball.} \end{matrix}$$

Lower Bound Each Voronoi cell is contained in ball of radius 2. The balls cover the box.



$$\Rightarrow m \geq \frac{k^2}{4\pi}$$

$$\text{So, } \frac{k^2}{4\pi} \leq m \leq \frac{k^2}{\pi}$$

Factor of 4 difference. Why would we care?

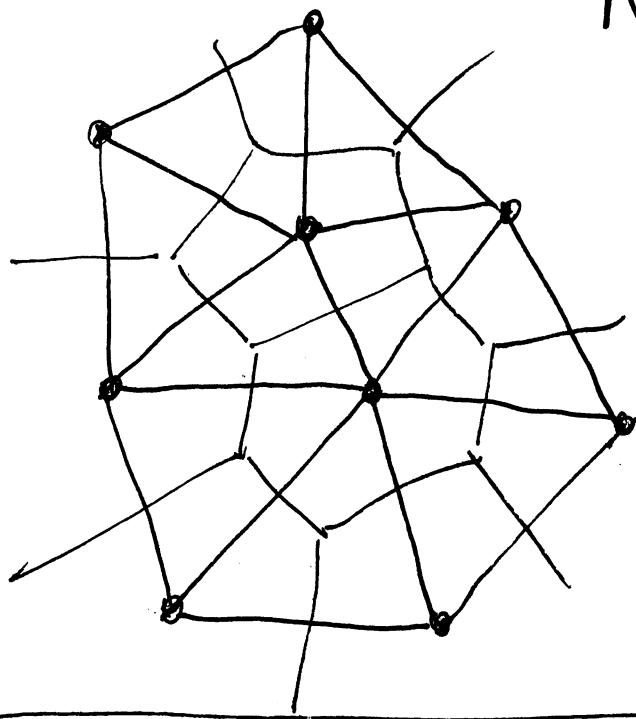
Because $\underline{4=2^d}$.

What does this have to do with meshing?

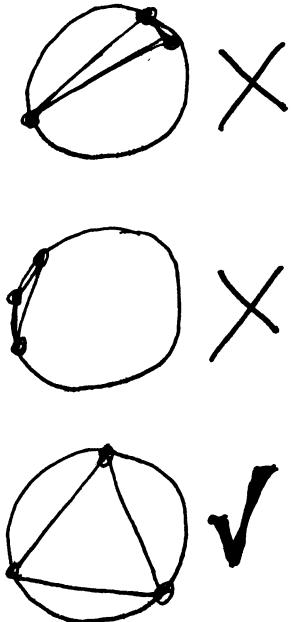
Good aspect ratio Voronoi Diagrams



Good radius-edge Delaunay Triangulation

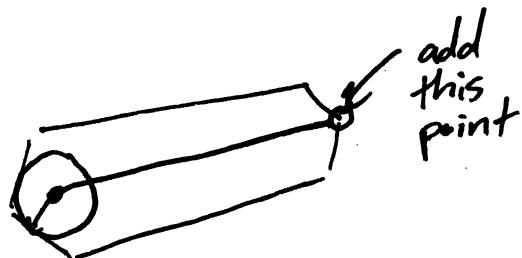


a standard measure of mesh quality.



Voronoi Refinement Meshing

If a Voronoi cell has bad aspect ratio,
add its farthest corner.

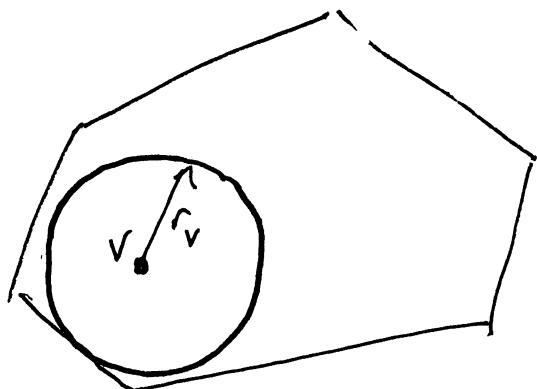


The local feature size

$f(x) = \text{distance to } \underline{\text{second}}\text{-nearest}$
input point. $\{f(x) > 0\}$

Key fact: Let $P \subset \mathbb{R}^d$ be the input
Let $M \triangleright P$ be the output.

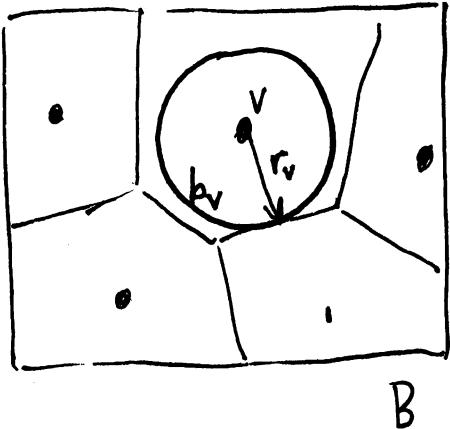
$\forall v \in M \quad f(v) < kr_v$ for some
constant k , where r_v is the
radius of the in-ball of $\text{Vor}(v)$.



k depends on what
we consider to be
a "bad" aspect ratio.

The Upper Bound

Recall: $\forall v \in M \quad f(v) < kr_v$



$$\text{Thm} \quad |M| < \frac{(k+1)^d}{\Gamma_d} \int_B \frac{dx}{f(x)^d}$$

↑
volume of a unit d-ball.

pf

$$\int_B \frac{dx}{f(x)^d} = \sum_{v \in M} \int_{b_v} \frac{dx}{f(x)^d} \quad [f(x) \geq 0]$$

$$> \sum_{v \in M} \int_{b_v} \frac{dx}{(k+1)r_v^d} \quad [f(x) < (k+1)r_v]$$

$$> \sum_{v \in M} \int_{b_v} \frac{dx}{((k+1)r_v)^d}$$

$$= \sum_{v \in M} \frac{r_v^d \Gamma_d}{r_v^d (k+1)^d}$$

$$= |M| \frac{\Gamma_d}{(k+1)^d}$$

Claim: $\forall x \in b_v, \quad f(x) < (k+1)r_v$

$$\begin{aligned} \text{pf} \quad f(x) &\leq f(v) + |x-v| \\ &\leq f(v) + r_v \\ &< (k+1)r_v. \end{aligned}$$

Lower Bound

Good Aspect Ratio
 $\Rightarrow \text{Vor}(v) < kb_v$

Easy: $\forall x \in \text{Vor}(v), f(x) \geq r_v$
... why?

Ihm

$$|M| > \frac{1}{k^d \Gamma_d} \int_B \frac{dx}{f(x)^d}$$

pf

$$\int_B \frac{dx}{f(x)^d} = \sum_{v \in M} \int_{\text{Vor}(v)} \frac{dx}{f(x)^d}$$

$$[f(x) \geq r_v]$$

$$\leq \sum_{v \in M} \int_{\text{Vor}(v)} \frac{dx}{r_v^d}$$

$$[\text{Vor}(v) < kb_v]$$

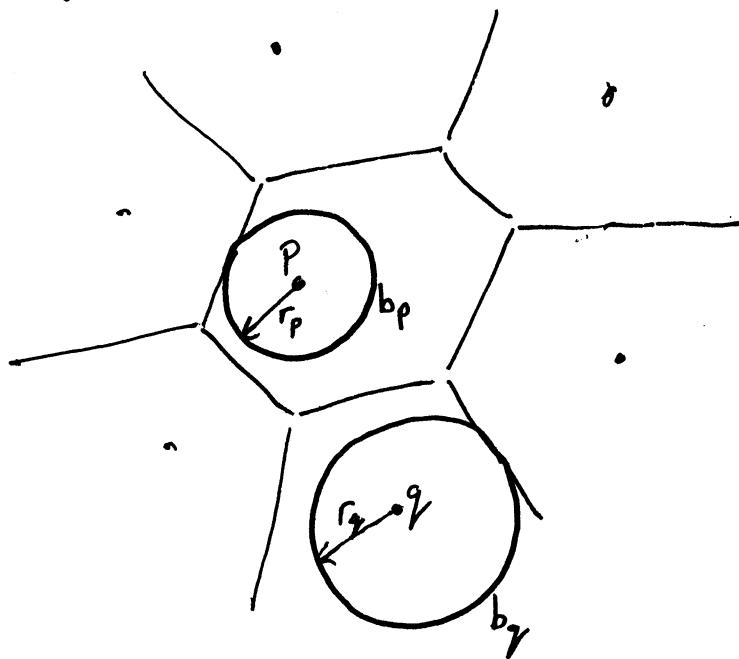
$$< \sum_{v \in M} \int_{kb_v} \frac{dx}{r_v^d}$$

$$= \sum_{v \in M} \frac{k^d r_v^d \Gamma_d}{r_v^d}$$

$$= |M| k^d \Gamma_d$$

■

Degree Bounds



$$\text{Claim 1: } |p-q| \leq 2k r_p$$

$$\text{Claim 2: } \frac{r_p}{k} \leq r_q \leq k r_p$$

$$\text{Claim 3: } b_q \subset 3k b_p$$

$$|Q| \left(\frac{1}{k} \right)^d r_p^d \prod_d = \sum_{q \in Q} \text{vol} \left(\frac{b_p}{k} \right) < \sum_{q \in Q} \text{vol}(b_q) < \text{vol}(3k b_p) = (3k)^d r_p^d \prod_d$$

$$\Rightarrow |Q| < 3^d$$

Note: Simple Exponential

Total Voronoi Edges < $3^d |M|$