Curve and Surface Reconstruction: Algorithms with Mathematical Analysis

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Curve Reconstruction

The simplest class of manifolds that pose nontrivial reconstruction problems are curves in the plane. We will describe two algorithms for curve reconstruction, CRUST and NN-CRUST in this chapter. First, we will develop some general results that will be applied to prove the correctness of the both algorithms.

A single curve in the plane is defined by a map $\xi : [0, 1] \to \mathbb{R}^2$ where $[0, 1]$ is the closed interval between 0 and 1 on the real line. The function ξ is one-to-one everywhere except at the endpoints where $\xi(0) = \xi(1)$. The curve is C^1-*smooth* if ξ has a continuous nonzero first derivative in the interior of $[0, 1]$ and the right derivative at 0 is same as the left derivative at 1 both being nonzero. If ξ has continuous ith derivatives, $i \geq 1$, at each point as well, the curve is called C^i-smooth. When we refer to a curve Σ in the plane, we actually mean the image of one or more such maps. By definition Σ does not self-intersect though it can have multiple components each of which is a closed curve, that is, without any endpoint.

For a finite sample to be a ε-sample for some $\varepsilon > 0$, it is essential that the local feature size f is strictly positive everywhere. While this is true for all C^2-smooth curves, there are C^1-smooth curves with zero local feature size at some point. For example, consider the curve

$$y = |x|^{\frac{4}{3}} \quad \text{for } -1 \leq x \leq 1$$

and join the endpoints $(-1, 1)$ and $(1, 1)$ with a smooth curve. This curve is C^1-smooth at $(0, 0)$ and its medial axis passes through the point $(0, 0)$. Therefore, the local feature size is zero at $(0, 0)$.

We learnt that C^1-smooth curves do not necessarily have positive minimum local feature size while C^2-smooth curves do. Are there curves in between C^1- and C^2-smooth classes with positive local feature size everywhere? Indeed, there is a class called $C^{1,1}$-smooth curves with this property. These curves are C^1-smooth and have normals satisfying a Lipschitz continuity property. To

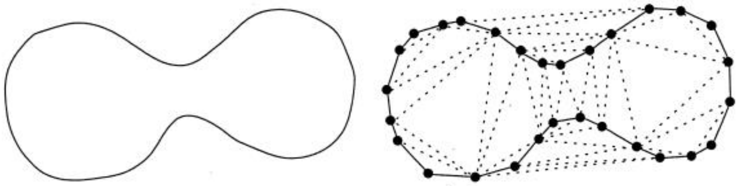


Figure 2.1. (a) A smooth curve and (b) its reconstruction from a sample shown with solid edges.

avoid confusions about narrowing down the class, we explicitly assume that Σ has strictly positive local feature size everywhere.

For any two points x, y in Σ define two curve segments, $\gamma(x, y)$ and $\gamma'(x, y)$ between x and y, that is, $\Sigma = \gamma(x, y) \cup \gamma'(x, y)$ and $\gamma(x, y) \cap \gamma'(x, y) = \{x, y\}$. Let P be a set of sample points from Σ. We say a curve segment is *empty* if its interior does not contain any point from P. An edge connecting two sample points, say p and q, is called *correct* if either $\gamma(p, q)$ or $\gamma'(p, q)$ is empty. In other words, p and q are two consecutive sample points on Σ. Any edge that is not correct is called *incorrect*. The goal of *curve reconstruction* is to compute a piecewise linear curve consisting of all and only correct edges. In Figure 2.1(b), all solid edges are correct and dotted edges are incorrect.

We will describe CRUST in Subsection 2.2 and NN-CRUST in Subsection 2.3. Some general results are presented in Subsection 2.1 which are used later to claim the correctness of the algorithms.

2.1 Consequences of ε-Sampling

Let P be a ε-sample of Σ. For sufficiently small $\varepsilon > 0$, several properties can be proved.

Lemma 2.1 (Empty Segment). *Let $p \in P$ and $x \in \Sigma$ so that $\gamma(p, x)$ is empty. Let the perpendicular bisector of px intersect the empty segment $\gamma(p, x)$ at z. If $\varepsilon < 1$ then*

(i) the ball $B_{z,\|p-z\|}$ intersects Σ only in $\gamma(p, x)$,
(ii) the ball $B_{z,\|p-z\|}$ is empty, and
(iii) $\|p - z\| \leq \varepsilon f(z)$.

Proof. Let $B = B_{z,\|p-z\|}$ and $\gamma = \gamma(p, x)$. Suppose $B \cap \Sigma \neq \gamma$ (see Figure 2.2). Shrink B continuously centering z until Int $B \cap \Sigma$ becomes a

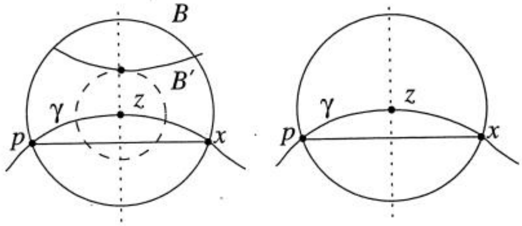


Figure 2.2. Illustration for the Empty Segment Lemma 2.1. The picture on the left is impossible while the one on the right is correct.

1-ball and it is tangent to some other point of Σ. Let B' be the shrunken ball. The ball B' exists as $B_{z,\delta} \cap \Sigma$ is a 1-ball for some sufficiently small $\delta > 0$ and $B \cap \Sigma$ is not a 1-ball. The ball B' is empty of any sample point as $\operatorname{Int} B'$ intersects Σ only in a subset of γ which is empty. But, since $B' \cap \Sigma$ is not a 1-ball, it contains a medial axis point by the Feature Ball Lemma 1.1. Thus, its radius is at least $f(z)$. The point z does not have any sample point within $f(z)$ distance as B' is empty. This contradicts that P is a ε-sample of Σ where $\varepsilon < 1$. Therefore, B intersects Σ only in $\gamma(p, x)$ completing the proof of (i).

Property (ii) follows immediately as $\gamma(p, x)$ is empty and B intersects Σ only in $\gamma(p, x)$. By ε-sampling, the nearest sample point p to z is within $\varepsilon f(z)$ distance establishing (iii). ■

The Empty Segment Lemma 2.1 implies that points in an empty segment are close and any correct edge is Delaunay when ε is small.

Lemma 2.2 (Small Segment). *Let x, y be any two points so that $\gamma(x, y)$ is empty. Then $\|x - y\| \leq \frac{2\epsilon}{1-\epsilon} f(x)$ for $\varepsilon < 1$.*

Proof. Since $\gamma(x, y)$ is empty, it is a subset of an empty segment $\gamma(p, q)$ for two sample points p and q. Let z be the point where the perpendicular bisector of pq meets $\gamma(p, q)$. Consider the ball $B = B_{z,\|p-z\|}$. Since $\gamma(p, q)$ is empty, the ball B has the properties stated in the Empty Segment Lemma 2.1. Since B contains $\gamma(p, q)$, both x and y are in B. Therefore, $\|z - x\| \leq \varepsilon f(z)$ by the ε-sampling condition. By the Feature Translation Lemma 1.3 $f(z) \leq \frac{f(x)}{1-\varepsilon}$. We have

$$\begin{aligned}\|x - y\| \leq 2\|p - z\| &\leq 2\varepsilon f(z) \\ &\leq \frac{2\varepsilon}{1-\varepsilon} f(x).\end{aligned}$$

■

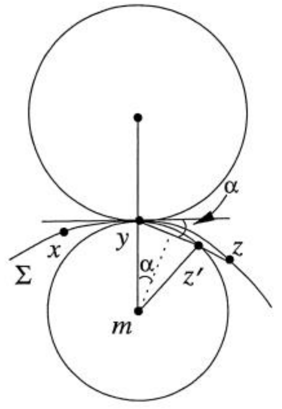


Figure 2.3. Illustration for the Segment Angle Lemma 2.4.

Lemma 2.3 (Small Edge). *Let pq be a correct edge. For $\varepsilon < 1$,*

(i) $\|p - q\| \le \frac{2\varepsilon}{1-\varepsilon} f(p)$ *and*
(ii) pq *is Delaunay.*

Proof. Any correct edge pq has the property that either $\gamma(q, p)$ or $\gamma(p, q)$ is empty. Therefore, (i) is immediate from the Small Segment Lemma 2.2. It follows from property (ii) of the Empty Segment Lemma 2.1 that there exists an empty ball circumscribing the correct edge pq proving (ii). ■

If three points x, y, and z on Σ are sufficiently close, the segments xy and yz make small angles with the tangent at y. This implies that the angle $\angle xyz$ is close to π. As a corollary two adjacent correct edges make an angle close to π.

Lemma 2.4 (Segment Angle). *Let x, y, and z be three points on Σ with $\|x - y\|$ and $\|y - z\|$ being no more than $\frac{2\varepsilon}{1-\varepsilon} f(y)$ for $\varepsilon < \frac{1}{2}$. Let α be the angle between the tangent to Σ at y and the line segment yz. One has*

(i) $\alpha \le \arcsin \frac{\varepsilon}{1-\varepsilon}$ *and*
(ii) $\angle xyz \ge \pi - 2 \arcsin \frac{\varepsilon}{1-\varepsilon}$.

Proof. Consider the two medial balls sandwiching Σ at y as in Figure 2.3. Let α be the angle between the tangent at y and the segment yz. Since z lies outside the medial balls, the length of the segment yz' is no more than that of yz where z' is the point of intersection of yz and a medial ball as shown.

In that case,

$$\alpha \leq \arcsin\left(\left(\frac{\|y - z'\|}{2}\right) / (\|m - y\|)\right)$$
$$= \arcsin\left(\left(\frac{\|y - z\|}{2}\right) / (\|m - y\|)\right).$$

It is given that $\|y - z\| \leq \frac{2\varepsilon}{1-\varepsilon} f(y)$ where $\varepsilon < \frac{1}{2}$. Also, $\|m - y\| \geq f(y)$ since m is a medial axis point. Plugging in these values we get

$$\alpha \leq \arcsin \frac{\varepsilon}{1 - \varepsilon}$$

completing the proof of (i). We have

$$\angle myz \geq \frac{\pi}{2} - \alpha$$
$$\angle myz \geq \frac{\pi}{2} - \arcsin \frac{\varepsilon}{1 - \varepsilon}.$$

Similarly, it can be shown that $\angle myx \geq \frac{\pi}{2} - \arcsin \frac{\varepsilon}{1-\varepsilon}$. Property (ii) follows immediately as $\angle xyz = \angle myz + \angle myx$. ■

Since any correct edge pq has a length no more than $\frac{2\varepsilon}{1-\varepsilon} f(p)$ for $\varepsilon < 1$ (Small Edge Lemma 2.3), we have the following result.

Lemma 2.5 (Edge Angle). *Let pq and pr be two correct edges incident to p. We have $\angle qpr \geq \pi - 2 \arcsin \frac{\varepsilon}{1-\varepsilon}$ for $\varepsilon < \frac{1}{2}$.*

2.2 Crust

We have already seen that all correct edges connecting consecutive sample points in a ε-sample are present in the Delaunay triangulation of the sample points if $\varepsilon < 1$. The main algorithmic challenge is to distinguish these edges from the rest of the Delaunay edges. The Crust algorithm achieves this by observing some properties of the Voronoi vertices.

2.2.1 Algorithm

Consider Figure 2.4. The left picture shows the Voronoi diagram clipped within a box for a dense sample of a curve. The picture on the right shows the Voronoi vertices separately. A careful observation reveals that the Voronoi vertices lie near the medial axis of the curve (Exercise 8). The Crust algorithm exploits this fact. All empty balls circumscribing incorrect edges in Del P cross the medial axis and hence contain Voronoi vertices inside. Therefore, they cannot appear

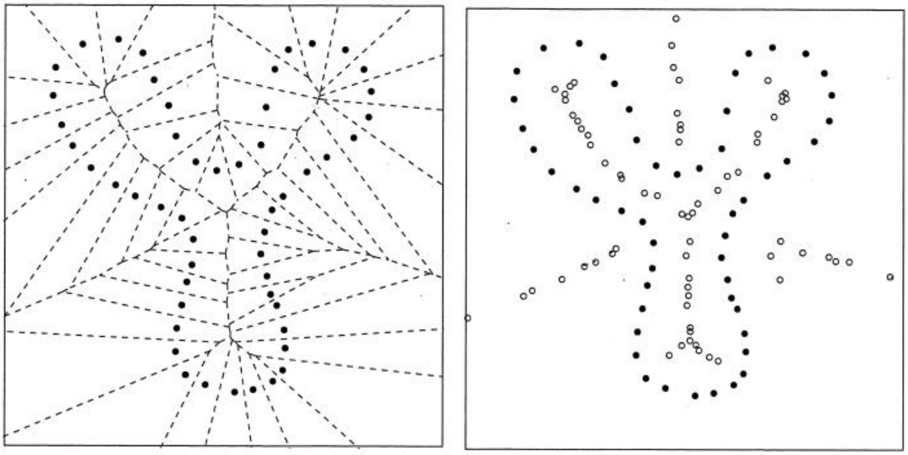


Figure 2.4. Voronoi vertices approximate the medial axis of a curve in the plane. The Voronoi vertices are shown with hollow circles in the right picture.

in the Delaunay triangulation of $P \cup V$ where V is the set of Voronoi vertices in Vor P. On the other hand, all correct edges still survive in Del $(P \cup V)$. So, the algorithm first computes Vor P and then computes the Delaunay triangulation of $P \cup V$ where V is the set of Voronoi vertices of Vor P. The Delaunay edges of Del $(P \cup V)$ that connect two points in P are output. It is proved that an edge is output if and only if it is correct.

CRUST(P)

```
1  compute Vor P;
2  let V be the Voronoi vertices of Vor P;
3  compute Del (P ∪ V);
4  E := ∅;
5  for each edge pq ∈ Del(P ∪ V) do
6     if p ∈ P and q ∈ P
7        E := E ∪ pq;
8     endif
9  output E.
```

The Voronoi and the Delaunay diagrams of a set of n points in the plane can be computed in $O(n \log n)$ time and $O(n)$ space. The second Delaunay triangulation in step 3 deals with $O(n)$ points as the Voronoi diagram of n points can have at most $2n$ Voronoi vertices. Therefore, CRUST runs in $O(n \log n)$ time and takes $O(n)$ space.

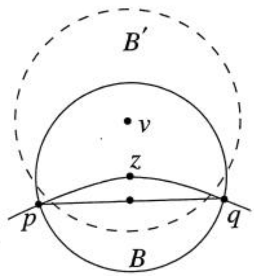


Figure 2.5. Illustration for the Correct Edge Lemma 2.6.

2.2.2 Correctness

The correctness of CRUST is proved in two parts. First, it is shown that each correct edge is present in the output of CRUST (Correct Edge Lemma 2.6). Then, it is shown that no incorrect edge is output (Incorrect Edge Lemma 2.7).

Lemma 2.6 (Correct Edge). *Each correct edge is output by* CRUST *when* $\varepsilon < \frac{1}{5}$.

Proof. Let pq be a correct edge. Let z be the point where the perpendicular bisector of pq intersects the empty segment $\gamma(p, q)$. Consider the ball $B = B_{z,\|p-z\|}$. This ball is empty of any point from P when $\varepsilon < 1$ (Empty Segment Lemma 2.1 (i)). We show that this ball does not contain any Voronoi vertex of Vor$P$ either.

Suppose that B contains a Voronoi vertex, say v, from V (Figure 2.5). Then by simple circle geometry the maximum distance of v from p is $2\|p - z\|$. Thus, $\|p - v\| \le 2\|p - z\|$. Since $\|p - z\| \le \varepsilon f(z)$ by the Empty Segment Lemma 2.1(iii), we have

$$\|p - v\| \le 2\varepsilon f(z) \le \frac{2\varepsilon}{1-\varepsilon} f(p).$$

The Delaunay ball B' centering v contains three points from P on its boundary. This means $\mathrm{bd}B' \cap \Sigma$ is not a 0-sphere. So, B' contains a medial axis point by the Feature Ball Lemma 1.1. As the Delaunay ball B' is empty, p cannot lie in Int B'. So, the medial axis point in B' lies within $2\|p - v\|$ distance from p. Therefore, $2\|p - v\| \ge f(p)$. But, $\|p - v\| \le \frac{2\varepsilon}{1-\varepsilon} f(p)$ enabling us to reach a contradiction when $\frac{2\varepsilon}{1-\varepsilon} < \frac{1}{2}$, that is, when $\varepsilon < \frac{1}{5}$.

Therefore, for $\varepsilon < \frac{1}{5}$, there is a circumscribing ball of pq empty of any point from $P \cup V$. So, it appears in Del $(P \cup V)$ and is output by CRUST as it connects two points from P. ∎

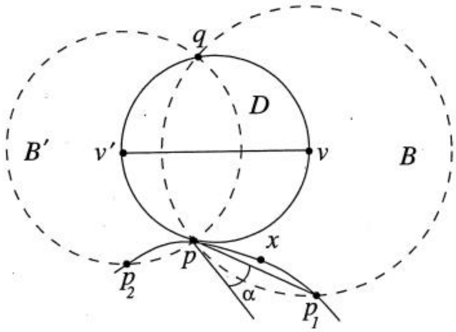


Figure 2.6. Illustration for the Incorrect Edge Lemma 2.7.

Lemma 2.7 (Incorrect Edge). *No incorrect edge is output by* CRUST *when* $\varepsilon < 1/5$.

Proof. We need to show that there is no ball, empty of both sample points and Voronoi vertices, circumscribing an incorrect edge between two sample points, say p and q. For the sake of contradiction, assume that D is such a ball.

Let v and v' be the two points where the perpendicular bisector of pq intersects the boundary of D (see Figure 2.6). Consider the two balls $B = B_{v,r}$ and $B' = B_{v',r'}$ that circumscribe pq.

We claim that both B and B' are empty of any sample points. Suppose on the contrary, any one of them, say B, contains a sample point. Then, one can push D continually toward B by moving its center on the perpendicular bisector of pq and keeping p, q on its boundary. During this motion, the deformed D would hit a sample point s for the first time before its center reaches v. At that moment p, q, and s define a ball empty of any other sample points. The center of this ball is a Voronoi vertex in Vor$P$ which resides inside D. This is a contradiction as D is empty of any Voronoi vertex from V.

The angle $\angle vpv'$ is $\pi/2$ as vv' is a diameter of D. The tangents to the boundary circles of B and B' at p are perpendicular to vp and $v'p$ respectively. Therefore, the tangents make an angle of $\pi/2$. This implies that Σ cannot be tangent to both B and B' at p.

First, consider the case where Σ is tangent neither to B nor to B' at p. Let p_1 and p_2 be the points of intersection of Σ with the boundaries of B and B' respectively that are consecutive to p among all such intersections. Our goal will be to show that either the curve segment pp_1 or the curve segment pp_2 intersects B or B' rather deeply and thereby contributing a long empty segment which is prohibited by the sampling condition.

The curve segment between p and p_1 and the curve segment between p and p_2 do not have any sample point other than p. By the Small Segment Lemma 2.2 both $\|p - p_1\|$ and $\|p - p_2\|$ are no more than $\frac{2\varepsilon}{1-\varepsilon} f(p)$ for $\varepsilon < \frac{1}{5}$. So by the Segment Angle Lemma 2.4, $\angle p_1 p p_2 \leq \pi - 2 \arcsin \frac{2\varepsilon}{1-\varepsilon}$.

Without loss of generality, let the angle between pp_1 and the tangent to B at p be larger than the angle between pp_2 and the tangent to B' at p. Then, pp_1 makes an angle α with the tangent to B at p where

$$
\begin{aligned}
\alpha &\geq \frac{1}{2}\left(\left(\pi - 2 \arcsin \frac{\varepsilon}{1-\varepsilon}\right) - \frac{\pi}{2}\right) \\
&= \frac{\pi}{4} - \arcsin \frac{\varepsilon}{1-\varepsilon}.
\end{aligned}
$$

Consider the other case where Σ is tangent to one of the two balls B and B' at p. Without loss of generality, assume that it is tangent to B' at p. Again the lower bound on the angle α as stated above holds.

Let x be the point where the perpendicular bisector of pp_1 intersects the curve segment between p and p_1. Clearly, x is in B. Since B intersects Σ at p and q which are not consecutive sample points, it cannot contain $\gamma(p, q)$ or $\gamma'(p, q)$ inside completely. This means $B \cap \Sigma$ cannot be a 1-ball. So, by the Feature Ball Lemma 1.1, B has a medial axis point and thus its radius r is at least $f(x)/2$. By simple geometry, one gets that

$$
\begin{aligned}
\|p - x\| &\geq \frac{1}{2}\|p - p_1\| \\
&= r \sin\alpha \\
&\geq \frac{1}{2} f(x) \sin\alpha.
\end{aligned}
$$

By property (iii) of the Empty Segment Lemma 2.1 $\|p - x\| \leq \varepsilon f(x)$. We reach a contradiction if

$$
2\varepsilon < \sin\left(\frac{\pi}{4} - \arcsin \frac{\varepsilon}{1-\varepsilon}\right).
$$

For $\varepsilon < \frac{1}{5}$, this inequality is satisfied. ■

Combining the Correct Edge Lemma 2.6 and the Incorrect Edge Lemma 2.7 we get the following theorem.

Theorem 2.1. *For $\varepsilon < \frac{1}{5}$,* CRUST *outputs all and only correct edges.*

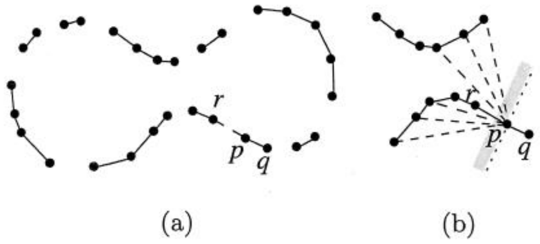


Figure 2.7. (a) Only nearest neighbor edges may not reconstruct a curve and (b) half neighbor edges such as pr fill up the gaps.

2.3 NN-Crust

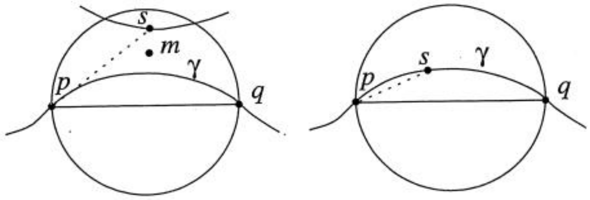
The next algorithm for curve reconstruction is based on the concept of nearest neighbors. A point $p \in P$ is a nearest neighbor of $q \in P$ if there is no other point $s \in P \setminus \{p, q\}$ with $\|q - s\| < \|q - p\|$. Notice that p being a nearest neighbor of q does not necessarily mean that q is a nearest neighbor of p.

We first observe that edges that connect nearest neighbors in P must be correct edges if P is sufficiently dense. But, all correct edges do not connect nearest neighbors. Figure 2.7 shows all edges that connect nearest neighbors. The missing correct edges in this example connect points that are not nearest neighbors. However, these correct edges connect points that are not very far from being nearest neighbors. We capture them in NN-CRUST using the notion of *half neighbors*.

2.3.1 Algorithm

Let pq be an edge connecting p to its nearest neighbor q and $\overrightarrow{pq}$ be the vector from p to q. Consider the closed half-plane H bounded by the line passing through p with $\overrightarrow{pq}$ as outward normal. Clearly, $q \notin H$. The nearest neighbor to p in the set $H \cap P$ is called its *half neighbor*. In Figure 2.7(b), r is the half neighbor of p. It can be shown that two correct edges incident to a sample point connect it to its nearest and half neighbors.

The above discussion immediately suggests an algorithm for curve reconstruction. But, we need efficient algorithms to compute nearest neighbor and half neighbor for each sample point. The Delaunay triangulation $\operatorname{Del} P$ turns out to be useful for this computation as all correct edges are Delaunay if P is sufficiently dense. The Small Edge Lemma 2.3 implies that, for each sample point p, it is sufficient to check only the Delaunay edges to determine correct edges. We check all edges incident to p in Del P and determine the shortest edge connecting it to its nearest neighbor, say q. Next, we check all other edges incident to p which make at least $\frac{\pi}{2}$ angle with pq at p and choose the shortest



(a) (b)

Figure 2.8. Diametric ball of pq intersects Σ in (a) two components and (b) single component.

among them. This second edge connects p to its half neighbor. The entire computation can be done in time proportional to the number of edges incident to p. Since the sum of the number of incident edges over all vertices in the Delaunay triangulation is $O(n)$ where $|P| = n$, correct edge computation takes only $O(n)$ time once Del P is computed. The Delaunay triangulation of a set of n points in the plane can be computed in $O(n \log n)$ time which implies that NN-crust takes $O(n \log n)$ time.

NN-CRUST(P)

```
compute Del P;
E = ∅;
for each p ∈ P do
   compute the shortest edge pq in Del P;
   compute the shortest edge ps so that ∠pqs ≥ π/2;
   E = E ∪ {pq, ps};
endfor
output E.
```

2.3.2 Correctness

As we discussed before, NN-CRUST computes edges connecting each sample point to its nearest and half neighbors. The correctness of NN-CRUST follows from the proofs that these edges are correct.

Lemma 2.8 (Neighbor). *Let $p \in P$ be any sample point and q be its nearest neighbor. The edge pq is correct for $\varepsilon < \frac{1}{3}$.*

Proof. Consider the ball B with pq as diameter. If B does not intersect Σ in a 1-ball, it contains a medial axis point by the Feature Ball Lemma 1.1 (see Figure 2.8(a)). This means $\|p - q\| > f(p)$. A correct edge ps satisfies

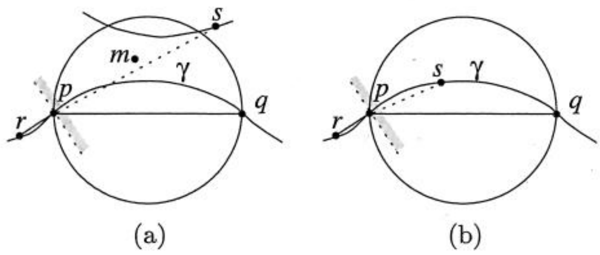


Figure 2.9. Diametric ball of pq intersects Σ in (a) more than one component and (b) a single component.

$\|p - s\| \leq \frac{2\varepsilon}{1-\varepsilon} f(p)$ by the Small Edge Lemma 2.3. Thus, for $\varepsilon < \frac{1}{3}$ we have $\|p - s\| < \|p - q\|$, a contradiction to the fact that q is the nearest neighbor to p.

So, B intersects Σ in a 1-ball, namely $\gamma = \gamma(p, q)$ as shown in Figure 2.8(b). If pq is not correct, γ contains a sample point, say s, between p and q inside B. Again, we reach a contradiction as $\|p - s\| < \|p - q\|$. ■

Next we show that edges connecting a sample point to its half neighbors are also correct.

Lemma 2.9 (Half Neighbor). *An edge pq where q is a half neighbor of p is correct when $\varepsilon < \frac{1}{3}$.*

Proof. Let r be the nearest neighbor of p. According to the definition $\overrightarrow{pq}$ makes at least $\frac{\pi}{2}$ angle with $\overrightarrow{pr}$.

If pq is not correct, consider the correct edge ps incident to p other than pr. By the Edge Angle Lemma 2.5, $\overrightarrow{ps}$ also makes at least $\frac{\pi}{2}$ angle with $\overrightarrow{pr}$ for $\varepsilon < 1/3$. We show that s is closer to p than q. This contradicts that q is the half neighbor of p since both $\overrightarrow{ps}$ and $\overrightarrow{pq}$ make an angle at least $\frac{\pi}{2}$ with $\overrightarrow{pr}$.

Consider the ball B with pq as a diameter. If B does not intersect Σ in a 1-ball (Figure 2.9(a)), it would contain a medial axis point and thus $\|p - q\| \geq f(p)$. On the other hand, for $\varepsilon < \frac{1}{3}$, $\|p - s\| \leq \frac{2\varepsilon}{1-\varepsilon} f(p)$ by the Small Edge Lemma 2.3. We get $\|p - s\| < \|p - q\|$ for $\varepsilon < \frac{1}{3}$ as required for contradiction. Next, assume that B intersects Σ in a 1-ball, namely in $\gamma(p, q)$, as in Figure 2.9(b). Since pq is not a correct edge, s must be on this curve segment. It implies $\|p - s\| < \|p - q\|$ as required for contradiction. ■

Theorem 2.2. NN-CRUST *computes all and only correct edges when $\varepsilon < \frac{1}{3}$.*

Proof. By the Small Edge Lemma 2.3 all correct edges are Delaunay. Steps 4 and 5 assure that all edges joining sample points to their nearest and half neighbors are computed as output. These edges are correct by the Neighbor Lemma 2.8 and the Half Neighbor Lemma 2.9 when $\varepsilon < \frac{1}{3}$. Also, there is no other correct edges since each sample point can only be incident to exactly two correct edges. ■

2.4 Notes and Exercises

In its simplest form the curve reconstruction problem appears in applications such as pattern recognition, image boundary detection, and cluster analysis. In the 1980s, several geometric graphs connecting a set of points in the plane were discovered which reveal a pattern among the points. The influence graph of Toussaint [11]; the β-skeleton of Kirkpatrick and Radke [62]; and the α-shapes of Edelsbrunner, Kirkpatrick, and Seidel [46] are such graphs.

Recall that a sample of a curve Σ is called globally *δ-uniform* if each point $x \in \Sigma$ has a sample point within a fixed distance δ. Several algorithms were devised to reconstruct curves from δ-uniform samples with δ being sufficiently small. Attali proposed a Delaunay-based reconstruction for such samples [9] (Exercise 3). de Figueiredo and de Miranda Gomes [27] showed that Euclidean minimum spanning tree (EMST) can reconstruct curves with boundaries from sufficiently dense uniform sample.

For a point set $P \subset \mathbb{R}^2$, let $\mathbb{N}$ denote the space of all points covered by open 2-balls of radius α around each point in P. The α-shape of P defined by Edelsbrunner, Kirkpatrick, and Seidel [46] is the underlying space of the restricted Delaunay triangulation Del $P|_{\mathbb{N}}$. Bernardini and Bajaj [12] proved that the α-shapes reconstruct curves from globally uniform samples that is sufficiently dense (Exercise 6).

The first breakthrough in reconstructing curves from nonuniform samples was made by Amenta, Bern, and Eppstein [5]. The presented CRUST algorithm is taken from this paper with some modifications in the proofs. Following the development of CRUST, Dey and Kumar devised the NN-CRUST algorithm [36]. The presented NN-CRUST algorithm is taken from this paper again with some modifications in the proofs. This algorithm also can reconstruct curves in three and higher dimensions, albeit with appropriate modifications of the proofs (Exercise 4).

The CRUST and NN-CRUST assume that the sample is derived from a smooth curve without boundaries. The questions of reconstructing nonsmooth curves and curves with boundaries have also been studied.

Giesen [54] showed that a fairly large class of nonsmooth curves can be reconstructed by Traveling Salesman Path (or Tour). A curve Σ is called *benign* if the left tangent and the right tangent exist at each point and make an angle less than π. Giesen proved that, a benign curve Σ can be reconstructed from a sufficiently dense uniform sample by the Traveling Salesman Path (or Tour) in case Σ has a boundary (or no boundary). The uniform sampling condition was later removed by Althaus and Mehlhorn [3], who also gave a polynomial time algorithm to compute the Traveling Salesman Path (or Tour) in the special case of curve reconstruction. The Traveling Salesman approach cannot handle curves with multiple components. Also, the sample points representing the boundaries need to be known a priori to choose between a path or a tour.

Dey, Mehlhorn, and Ramos [38] presented an algorithm named CONSERVATIVE CRUST that provably reconstructs smooth curves with boundaries. Any algorithm for handling curves with boundaries faces a dilemma when an input point set samples a curve without boundary densely and simultaneously samples densely another curve with boundary. This dilemma is resolved in CONSERVATIVE CRUST by a justification on the output. For any input point set P, the graph output by the algorithm is guaranteed to be the reconstruction of a smooth curve possibly with boundary for which P is a dense sample. The main idea of the algorithm is that an edge pq is output only if its diametric ball is empty of all Voronoi vertices in Vor P. The rationale behind this choice is that these edges are small enough with respect to local feature size of the sampled curve since the Voronoi vertices approximate the medial axis. With a sampling condition tailored to handle nonsmooth curves, Funke and Ramos [52] and Dey and Wenger [41] proposed algorithms to reconstruct nonsmooth curves. The algorithm of Funke and Ramos can handle boundaries as well.

Exercises

(The exercise numbers with the superscript h and o indicate *hard* and *open* questions respectively.)

1. Give an example of a point set P such that P is a 1-sample of two curves for which the correct reconstructions are different.
2. Given a $\frac{1}{4}$-sample P of a C^2-smooth curve, show that all correct edges are Gabriel in Del $(P \cup V)$ where V is the set of Voronoi vertices in Vor P.
3. Let P be a ε-sample of a C^2-smooth curve without boundary. Let η_{pq} be the sum of the angles opposite to pq in the two (or one if pq is a convex hull edge) triangles incident to pq in Del P. Prove that there is a ε for which pq is correct if and only if $\eta_{pq} < \pi$.
4. Show that the NN-CRUST algorithm can reconstruct curves in three dimensions from sufficiently dense samples.

5. The Correct Edge Lemma 2.6 is proved for $\varepsilon < \frac{1}{5}$. Show that it also holds for $\varepsilon \leq \frac{1}{5}$. Similarly, show that the Neighbor Lemma 2.8 and the Half Neighbor Lemma 2.9 hold for $\varepsilon \leq \frac{1}{3}$.

6^h. Establish conditions for α and δ to guarantee that an α-shape reconstructs a C^2-smooth curve in the plane from a globally δ-uniform sample.

7^o. Gold and Snoeyink [58] showed that the CRUST algorithm can be modified to guarantee a reconstruction with $\varepsilon < 0.42$. Althaus [2] showed that the NN-CRUST algorithm can be proved to reconstruct curves from ε-samples for $\varepsilon < 0.5$. Can this bound on ε be improved? What is the largest value of ε for which curves can be reconstructed from ε-samples?

8^h. Let $v \in V_p$ be a Voronoi vertex in the Voronoi diagram Vor P of a ε-sample P of a C^2-smooth curve Σ. Show that there exists a point m in the medial axis of Σ so that $\|m - v\| = \tilde{O}(\varepsilon) f(p)$ when ε is sufficiently small (see Section 1.2.3 for $\tilde{O}$ definition).