Delaunay Refinement

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2D only

Input: PSLG all angles > 60° (C)

Output: 1) 2D simplicial complex
        2) A refinement of the PSLG
        3) Delaunay
        4) no small angle
        5) constant time opt size

Def. \text{lls}(x) = \text{dist to 2nd nearest feature}.

Def. \text{f: } \mathbb{R}^d \rightarrow \mathbb{R} \text{ is } \alpha\text{-Lipschitz if } \forall p, q \in \mathbb{R}^d
\[ |f(p) - f(q)| \leq \alpha \text{ dist}(p, q) \]

Claim. \text{lfs is } 1\text{-Lipschitz i.e.}
\[ \text{lfs}(p) \leq \text{lfs}(q) + \text{dist}(p, q) \]
Algorithm

Def: Circumball of a simplex is min radius ball B with vertices on \partial B.

Def: P encroaches simplex S if p \in \text{Inter}(\text{Circumball } S)

S' encroaches on S if circumcenter(s') \in \text{Inter}(\text{Circumball } S)

Def: Segment is a subsegment of an edge of C.

Delaunay Refinement (G)

1) Add a bounding box to G
2) Compute Delaunay of G + Box
3) While
   1) A segment is encroach add circum-center
   2) A tri is skinny add circum-center.
Algorithm Details

Subroutine: Split(segs, S)

1) Add circumcenter of s to V & update DT(V)

2) Remove s from S

3) Add halves of s to S.
Delaunay Refinement \((\text{PSLG } G, \text{ angle } \theta)\)

Init 1) Add enclosing box to \(G\)
2) \(S = \text{edge}(G)\)
3) \(V = \text{vertex}(G)\)
4) \(T = \text{DT}(V)\)

While \(\exists \text{encroached seg or skinny tri} \) do

1) While \(\exists s \in S \ (s \text{ encroached}) \) split seg(s)

2) If \(t \) skinny \((\text{radius-edge} > \gamma)\) then \((\star)\)
   - If \(t \) encroaches on seg \(s\) then
   - split seg(s)
   - else split tri(t)

Return \(\text{DT}(V)\)
Def \( N_V(P) \) = nearest vertex in \( V \) at last time
\( P \) was considered for insertion

E.g. at step (X) \( P \) is Circumcenter (+) was considered but may not have been added.

Def \( \text{Containing Dimension of } P = \) min dim feature containing \( P \).

E.g. \( P \) is an input point then \( CD(P) = 0 \)
\( P \) interior to an edge \( CD(P) = 1 \)
\( P \) on a \( CD(P) = 2 \)

Lemma \( C_e \) constant to \( C_e \& C_t \) depending only on \( \alpha \) s.t.

1) If \( CD(P) = 0 \) then \( \text{Mfs}(P) \leq N_V(P) \)
2) If \( CD(P) = 1 \) then \( \text{Mfs}(P) \leq C_e N_V(P) \)
3) If \( CD(P) = 2 \) then \( \text{Mfs}(P) \leq C_t N_V(P) \)
\[ \text{Proof: Induction on execution time.} \]

**Case \( \text{CD}(P) = 0 \)**

\[ \text{lfs}_G(P) \leq \text{lfs}_V(P) \leq \text{NN}(P) \]

**Case \( \text{CD}(P) = 1 \)**

Let \( P \in E_P \) (input edge) & \( P = \text{circum center of segment } S_p \)

\( \forall p \in S_p \subseteq E_p \)

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\( S_p \) must have been encroached by some point a

Pick a \( \in \text{Ball}(S_p) \) as follows.

1. \( \exists a \in \text{Ball}(S_p) \) at time \( \text{splitseg}(S_p) \)
   set a to closest such point to p.
else let $a \in B(S_p)$ be circumcenter of
skinnier tri that yielded to \( p \).

\underline{Subcase} \hspace{1cm} CD(a) = 0

\begin{align*}
\text{ffs}(p) & \leq \text{dist}(p, a) < NN(p) \\
CD(a) & = 0 \\
\text{we need} & \quad 1 \leq C_e
\end{align*}

\underline{Subcase} \hspace{1cm} CD(a) = 1

Let $a \in E_a$ (input edge)

(lets assume input angles $\geq 90^0$)

Thus $E_a \cap E_b = \emptyset$

\begin{align*}
\text{ffs}(p) & \leq \text{dist}(p, E_a) \leq \text{dist}(p, a) = NN(p) \\
\text{we need} & \quad 1 \leq C_e
\end{align*}

\((60^0 \text{ case?})\)
**Subcase** $CD(a) = 2$

By induction $\text{fS}(a) \leq C + \text{NN}(a) \leq C + r'$

1) $\text{NN}(p) = r$ (will yield to $S_p$)
2) $a \in B(p, r') \& x, y \in B(a, r') \Rightarrow r' \leq \sqrt{2} r$

$\text{fS}(p) \leq \text{fS}(a) + \text{dist}(p, a) \leq C + r' + r$

$\leq C + \sqrt{2} r + r$

$\leq (\sqrt{2} C + 1) \text{NN}(p)$

\[ \text{hied} \quad 1 + \sqrt{2} C_t \leq C_e \]
**Case** \( CD(p) = 2 \)

\[ WLOG \ a \text{ added before } b. \]

**Subcase** \( CD(b) = 0 \)

\[ \text{lfs}(p) = r = \text{NN}(p) \]

\[ 1 \leq c_t \]

**Subcase** \( CD(b) = 1 \)

\[ (\text{induct}) \text{lfs}(b) \leq c_e \text{NN}(b) \leq c_e \text{dist}(b, a) \]

radius-edge \( p > \frac{r}{c_e} \) \( \text{ie} \ e < pr \)

\[ \text{lfs}(p) \leq \text{lfs}(b) + \text{dist}(p, b) \leq \text{lfs}(b) + r \]

\[ \leq c_e \text{dist}(b, a) + r \]

\[ \leq c_e pr + r = (c_e p + 1) r \]

\[ = (c_e p + 1) \text{NN}(p) \]

\[ (c_e p + 1) \leq c_t \]
Subcase $CD(b) = 2$

Same as last case but $C_e$ in $C_t$

$C_t + \rho + 1 \leq C_t$ in $\left( \frac{1}{1-\rho} \right) \leq C_t$

Own list of need conditions

$1 \leq C_e$, $\left( \frac{1}{1-\rho} \right) \leq C_t$

$1 - \sqrt{2} C_t \leq C_e$, $1 + \rho C_e \leq C_t$ (**)
\[ \text{feasible} \quad \sqrt{2}p < 1 \quad p < \frac{1}{\alpha} \]

Computing \( \alpha \)

\[ \alpha = \sin^{-1}(e/2) \]

\[ 20^\circ < \sin^{-1}\left(\frac{1}{2\sqrt{2}}\right) \]

**Thm** \( p \in \text{Output then} \) \( \text{lfs}(p) \leq (c_{e+1}) \text{NN}_{\text{output}}(p) \)

** Pf**

Let \( g \) be NN of \( p \) in output.

Case 1. \( p \) added after \( g \) then \( \text{lfs}(p) \leq c_{e} \text{NN}(p) \)

Case 2. \( g \) \( \not\in p \).

\[ \text{lfs}(p) \leq \text{lfs}(g) + \text{dist}(p, g) \leq c_{e} \text{NN}(g) + \text{dist}(p, g) \]

\[ \leq c_{e} \text{NN}(p) + \text{NN}(p) = (c_{e}+1) \text{NN}(p) \]
Thm. DR generates a mesh with at most
\[ C \int_{\mathcal{B}^n_{\text{ext}}} \frac{1}{\text{dfs}^2(x)} \, dA \] vertices.

**Proof** Let \( r_p = \frac{\text{dfs}(p)}{a(Ce+1)} \)

\[ \text{Note} \quad \text{Balls } \mathcal{B}(p, r_p) \text{ are disjoint.} \]

\[ \text{Not} \quad \max_{x \in \mathcal{B}_p} \text{dfs}(x) \leq \text{dfs}(p) + r_p \]

\[ \int_{\mathcal{B}_p} \frac{1}{\text{dfs}(p)} \, dA \geq \text{Area} \left( \mathcal{B}_p \right) \frac{1}{\left( \text{dfs}(p) + r_p \right)^2} \approx \frac{\pi r_p^2}{\left( 2(Ce+1) r_p + r_p \right)^2} \]

\[ \approx \frac{\pi}{(2(C+3))^2} \equiv C' \]
\[
\int \frac{1}{d \mathcal{F}^2(x)} \, dA = \sum_{\text{PEV}(D)} \int_{B_p} \frac{1}{d \mathcal{F}^2(x)} \, dA \\
\geq \sum_{\text{PEV}} \frac{1}{4} c' = \frac{1}{4} c' \left| \mathcal{V} \right|
\]