CG
11/4/08

Delaunay Refinement
Ruppert

2D only

Input: PSLG all angles > 60^0 (G)

Output: 1) 2D simplicial complex
2) A refinement of the PSLG
3) Delaunay
4) no small angle
5) constant time opt size

Def $lfs(x) \equiv$ dist to 2nd nearest feature.

Def $f: R^d \to R$ is α-Lipschitz if $\forall p, q \in R^d$

$$|f(p) - f(q)| \le \alpha \, dist(p, q)$$

Claim lfs is 1-Lipschitz i.e.

$$lfs(p) \le lfs(q) + dist(p, q)$$

Algorithm

Def Circumball of a simplex is min radius ball B with vertices on ∂B.

Def p encroaches simplex S if $p \in$ Inter(Circumball S)
S' encroaches on S if circumcenter(S') $\in$ Inter(Circumball S)

Def segment is a subsegment of an edge of G.

Delaunay Refinement (G)

1) Add a bounding box to G
2) Compute Delaunay of G + Box
3) While
   1) A segment is encroach add circum-center
   2) A tri is skinny add circum-center.

Algorithm Details

Subroutine: Split(seg s, S)

1) Add circumcenter of s to V & update DT(V)

2) a) remove s from S

   b) add halves of s to S.

Delaunay Refinement (PSLG G, angle α)

Init
1) Add bounding box to G
2) S = edges(G)
3) V = vertices(G)
4) T = DT(V)

While $\exists$ encroached seg or skinny tri do

1) While $\exists s \in S$ (s encroached) splitseg(s)
2) If t skinny (radius-edge > r/e) then (*)
   If t encroaches on seg s then
   splitseg(s)
   else splittri(t)

Return DT(V)

<u>Def</u> $NN(p) \equiv$ nearest vertex in V at last time p was considered for insertion

e.g. at step (*) $p \equiv$ circumcenter(t) was considered but may not have been added.

<u>Def</u> <u>Containing Dimension</u> of $p \equiv$ min dim feature containing p.

eg p is an input point then $CD(p)=0$

p <u>interior</u> to an edge $CD(p)=1$

o.w. $CD(p)=2$

<u>Lemma</u> $\exists$ constants C_e & C_t depending only on α s.t.

1) <u>If</u> $CD(p)=0$ <u>then</u> $lfs(p) \leq NN(p)$

2) If $CD(p)=1$ then $lfs(p) \leq C_e \, NN(p)$

3) If $CD(p)=2$ then $lfs(p) \leq C_t \, NN(p)$

pf Induction on execution time.

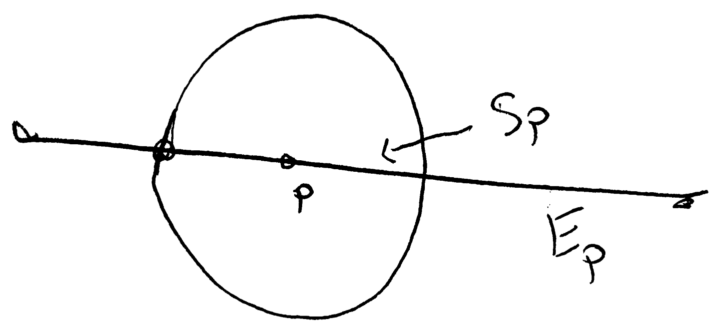
Case $CD(p)=0$

$$lfs_G(p) \leq lfs_V(p) \leq NN(p)$$

Case $CD(p)=1$

Let $p \in E_p$ (input edge) & p = circumcenter of segment S_p

ie $p \in S_p \subseteq E_p$



S_p must have been encroached by some point a

Pick $a \in Ball(S_p)$ as follows.

1) $\exists a \in Ball(S_p)$ at time splitseg(S_p)

set a to closest such point to p.

else let $a \in B(S_p)$ be circumcenter of skinny tri that yeilded to p.

Subcase $CD(a) = 0$

$$lfs(p) \underbrace{\leq}_{CD(a)=0} dist(p, a) \leq NN(p)$$

we need $\boxed{1 \leq C_e}$

Subcase $CD(a) = 1$

Let $a \in E_a$ (input edge)

(Lets assume input angles $\geq 90°$)

Thus $E_a \cap E_b = \emptyset$

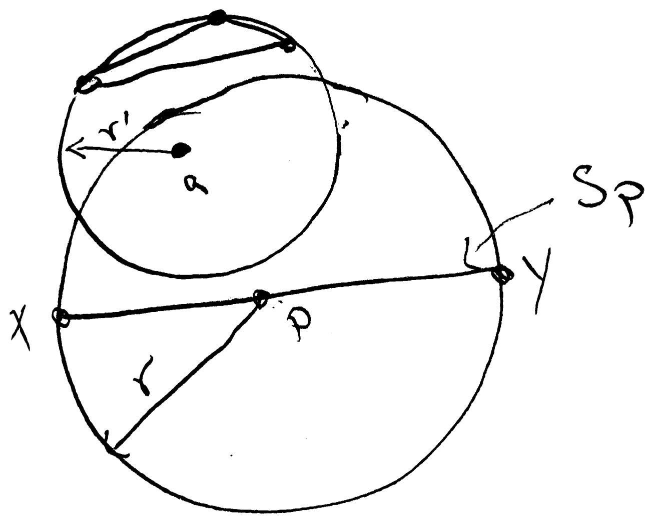
$$lfs(p) \leq dist(p, E_a) \leq dist(p, a) = NN(p)$$

$\boxed{1 \leq C_e}$

($60°$ case?)

Subcase $CD(a) = 2$

By induction $lfs(a) \le C_t\, NN(a) \le C_t r'$

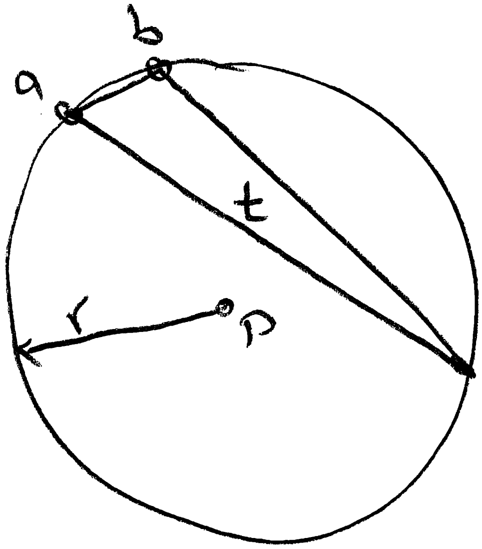


1) $NN(p) = r$ (a will yeild to S_p)

2) $a \in B(p, r)$ & $x, y \notin B(a, r') \Rightarrow r' \le \sqrt{2}\, r$

$$\begin{aligned} lfs(p) &\le lfs(a) + dist(p, a) \le C_t r' + r \\ &\le C_t \sqrt{2}\, r + r \\ &\le (\sqrt{2}\, C_t + 1)\, NN(p) \end{aligned}$$

need $\boxed{1 + \sqrt{2}\, C_t \le C_e}$



Case $CD(p)=2$

WLOG a added before b.

Subcase $CD(b)=0$

$lfs(p) = r = NN(p)$ $\boxed{1 \leq C_t}$

Subcase $CD(b)=1$

(induct) $lfs(b) \leq C_e NN(b) \leq C_e\, dist(b,a)$

$\boxed{\text{radius-edge } \rho > r/e \text{ ie } e < \rho r}$

$$lfs(p) \leq lfs(b) + dist(p,b) \leq lfs(b) + r$$
$$\leq C_e\, dist(b,a) + r$$
$$\leq C_e \rho r + r = (C_e \rho + 1) r$$
$$= (C_e \rho + 1) NN(p)$$

$$\boxed{(C_e \rho + 1) \leq C_t}$$

Subcase $CD(b) = 2$

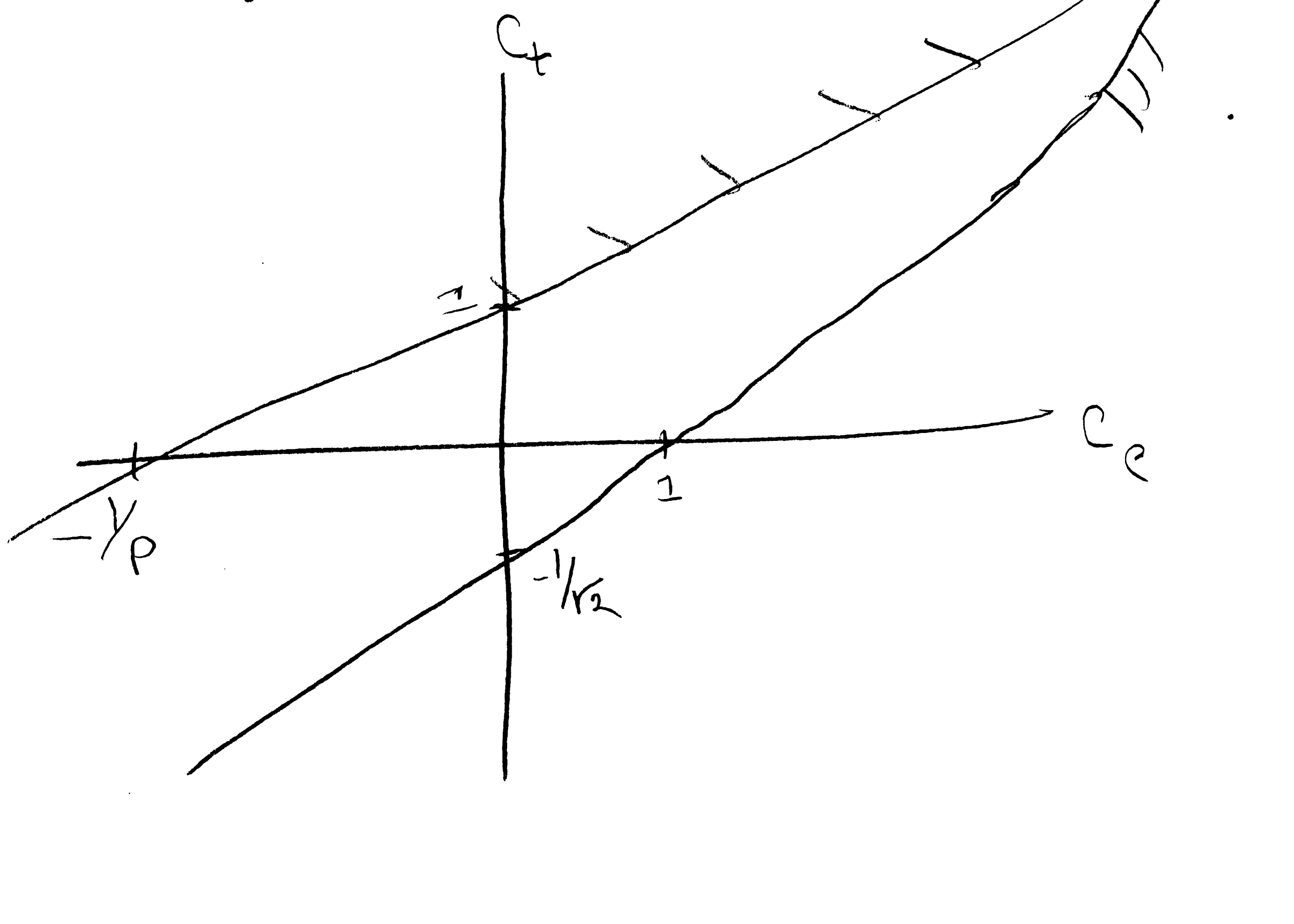
same as last case but c_e in C_t

$$(C_t \rho + 1) \le C_t \quad \text{ie} \quad \boxed{\left(\frac{1}{1-\rho}\right) \le C_t}$$

Our list of need conditions

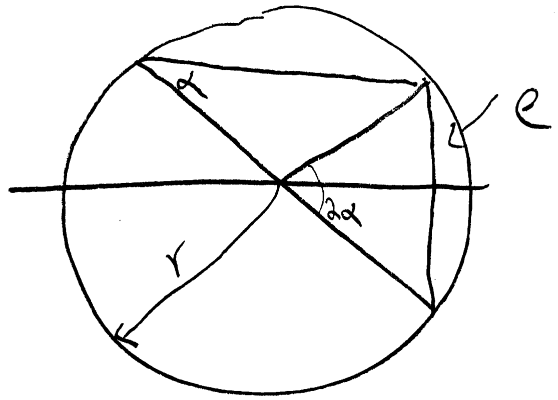
$$1 \le C_e, \quad \left(\frac{1}{1-\rho}\right) \le C_t$$

$$1 + \sqrt{2}\, C_t \le C_e \quad \& \quad (1 + \rho C_e) \le C_t \quad (**)$$



Feasible $\sqrt{2}\rho < 1$ $\quad \rho < 1/\sqrt{2}$

computing α



$\alpha = \sin^{-1}(e/2)$ $\quad 20^\circ = \sin^{-1}\left(\frac{1}{2\sqrt{2}}\right)$

Thm $p \in$ Output then $lfs(p) \leq (C_e+1)\, NN_{output}(p)$

pf

Let q be NN of p in output.

Case 1 p added after q then $lfs(p) \leq C_e\, NN(p)$

Case 2. q " " p.

$$lfs(p) \leq lfs(q) + dist(p,q) \leq C_e\, NN(q) + dist(p,q)$$

$$\leq C_e\, NN(p) + NN(p) = (C_e+1)\, NN(p)$$

<u>Thm</u> DR generates a mesh with at most

$$C\int_{B_{1x}} \frac{1}{lfs^2(x)}\, dA \text{ vertices.}$$

<u>pf</u> Let $r_p = \dfrac{lfs(p)}{2(C_e+1)}$

<u>Note</u> Balls $\underset{\underset{B_p}{\shortparallel}}{B(p, r_p)}$ are disjoint.

<u>Note</u> $\max_{x \in B_p} lfs(x) \leq lfs(p) + r_p$

$$\int_{B_p} \frac{1}{lfs^2(x)}\, dA \geq \text{Area}(B_p) \frac{1}{(lfs(p)+r_p)^2} = \frac{\pi r_p^2}{(2(C_e+1)r_p + r_p)^2}$$

$$= \frac{\pi}{(2C_e+3)^2} \equiv C'$$

$$\int_{Box} \frac{1}{lfs^2(x)}\, dA \geq \sum_{p \in V(D)} \frac{1}{4} \int_{B_p} \frac{1}{lfs^2(x)}\, dA$$

$$\geq \sum_{p \in V} \frac{1}{4} C' = \frac{1}{4} C' |V|$$