Mesh Generation

Goal: Partition domain into simplices.

∅

Simplex: vertex, segment, triangle, tetrahedron

Partition: Intersection of 2 simplices is a simplex

Conforming: to input

Well-shaped simplices

a) no small angles ≤ 0°

b) no large angles ≥ 180°

Small number of simplices (optimal size)
Aspect Ratio

\[ A(a, b, c) = \frac{\text{longest-side}}{\text{alt}} \]

\[ R(a, b, c) = \frac{\text{longest-side}}{\text{shortest-side}} \]

\[ 1 \]
\[ \text{Smallest-angle} \]

\[ \text{180}^\circ - \text{largest-angle} \]

radius-edge ratio = \( r/e \)

\( r = \) radius of circum sphere
\( e = \) shortest edge
Mesh Generation Methods

1) Quadtree (today)
2) Delaunay Refinement (Thursday)
3) Advancing Front
4) Ball-Packing

In 2D our input will be PSLG.
Simplex & Simplicial Complex

Def: \( P_0, \ldots, P_k \in \mathbb{R}^d \) are affinely independent if \( P_1 - P_0, \ldots, P_k - P_0 \) are independent.

Def: If \( P_0, \ldots, P_k \) are a-ind then \( CC(P_0, \ldots, P_k) \) is a \( k \)-simplex & \( \forall S \subseteq \{ P_0, \ldots, P_k \} \) \( CC(S) \) is a sub-simplex

Def: A set \( K \) of simplices in \( \mathbb{R}^d \) is a Simplicial Complex if:

1) \( K \) is closed under sub-simplex

2) \( S, T \in K \) then \( S \cap T \) sub-simplex of \( T \)

Def: \( \dim(K) = \max \{ \dim S | S \in K \} \)
Note: \( \text{PSLG} \) is a 1-dim simplicial complex in \( \mathbb{R}^2 \)

\( K \) & \( K' \) are simplicial complexes

**Def:** \( K' \) is a refinement of \( K \) if

\[ \forall s \in K \text{ of dim } k \exists s_1, \ldots, s_k \in K' \text{ of dim } k \]

\[ \text{st. } s = \bigcup_{i=1}^{k} s_i \]

**Input:** Simplicial complex \( K \) & Domain \( \mathcal{N} \)

\[ \forall s \in K \Rightarrow s \in \mathcal{N} \]

**Output:** refinement \( K' \) of \( K \) s.t.

\[ \bigcup_{s \in K'} s = \mathcal{N} \]
Quad-Tree Meshing

Input: set $X \subseteq \mathbb{R}^2$ of points $X \subseteq \mathcal{B}$ (box) $|X| = n$

Def $QT$ is a tree of nested square boxes.

The children of box $b$ are either

1) empty (leaf box)
2) 4 children of half the size (split of $b$)

Neighbors: 4 direct neighbors
8 extended neighbors
Def: QT is balanced if every leaf box has no side containing more than one interior node.

OK

not OK
Build-QT(\(X, B\))

\[\text{Init: } QT \ T = (X, B)\]

1) While \(X\) \text{ leaf box } (\(X', B\)) \text{ s.t. } B \text{ is crowded}\n
\quad \text{split } B \text{ and split } X'\n
2) Balance \(T\) by splitting.

3) Split all boxes containing a point \text{ has 8 extended neighbors (leaf boxes).}
Def: Let box $b$ be crowded if $\exists x \in b$
and one of the following holds:

1) $\exists y \neq x \in b$
2) $\exists y \in X$ s.t. $\text{dist}(x, y) \leq 2\sqrt{2} \cdot \text{side length}(b)$
3) An extended neighboring box $b$ is split.

**Warping**

$x \in b$ & $y$ closest corner of $b$ then warp $y$.

![Diagram of warping process]

$b$ empty & not warped

![Diagram of empty and not warped box]


Thm. If $T$ is a QT & $T'$ is its balanced version then $|T'| = O(|T|)$, $|T'| = \# \text{ boxes}$

\[ \text{RS} \]
\[ \text{note } T \text{ is a proper } k\text{-ary tree} \]
\[ \text{note } A \text{ proper } k\text{-ary tree with } i \text{ internal nodes has size } k\cdot i + 1 \text{ (induct)} \]

boxes of $T$ are old.

Claim. A new internal box has an extended neigh which is old.

Proof. by contraction
Let $b$ be the smallest internal new box with no old ext neigh.

$b_{\text{internal}} \Rightarrow$ a side of $b$ is split twice e

$\text{e.g.}$

\[ b \quad \quad \quad b' \]

$b'$ is new with no old neigh. contr.

\[ \#_{\text{int}}(T') \geq 8 \cdot n \]

\[ (T') \leq 4 \cdot \#(\text{int}(T')) + 1 \leq 32n + 1 \]

\[ \text{Thm } \text{Balanced QT can be computed in } O(dn) \]

time $d_1 = \text{depth } 0$