The Line Segment Intersection Prob 9/11/08

Input: A set of n line segments
Output: Report all intersections

Naive: \( O(n^2) \)
Goal: \( O(n \log n + |I|) \) Today \( O((n+|I|) \log n) \)

Worst case: \(|I| = \Omega(n^3)\)

Motivation: Map Overlay

Algorithm: sweep line
Optimal: Incremental
Map overlay prob

Segments: \( S = \{s_1, \ldots, s_n\} \) (no vertical seg)

Output: Break all segments into subseg s.t.

Two subseg intersect only at endpoints

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Sweep Line Alg

Let \( P = \) endpoints of seg.

\( l = \) horizontal line disjoint from \( PVI \).

\( l \) linearly orders \( S \).

The order only changes a \( S \).

Store order in Balanced BST.

Events \( \leq \) PVI

Idea: sweep \( x \) top to bottom stopping at events.
But we do not know I!

Compute events in I just-in-time.

Claim: If next event is the intersection of S & S', then S & S' are neighbors.

Priority Queue $Q_e$ of events

Inductively: $Q_e$ contains

1) $p$

2) Neighbors in $I$ below $I_0$. 


Handling Events

\[ U(P) = \text{subseg with upper end point } P \]
\[ L(P) = \text{"lower"} \]
\[ C(P) = \text{"intersection } P. \]

Procedure Handle Event(P point, T tree, Q queue)

1. Use \( C(P) \) form new subseg and add to \( U(P) \) \& \( L(P) \).
2. Use \( L(P) \) delete \((s, T)\)
3. Use \( U(P) \) insert \((s, T)\)
4. A new neighbor add intersection to \( Q \)

Let m # subseg

Alg runs in \( O(m \log n) \) Time

there are at most m delete/inserts into \( T \& Q \)
How many Subseg?

To show: \( \# \text{Subseg} = O(n + |E|) \)

**Embedded Planar Graph** \((G = (V, E), \varphi: G \rightarrow \mathbb{R}^2)\)

- \(\varphi(e) = \text{path}\)
- \(\varphi(e) \cap \varphi(e') = \text{only endpoints}\)

**Euler's Formula** \((G, \varphi)\)

\[ n_V = \# \text{Vertices} \]
\[ n_E = \# \text{Edges} \]
\[ n_f = \# \text{connected boundaries (faces)} \]
\[ c = \# \text{connected components} \]

Then \( n_f - n_e + n_V = 2c \)

\[ \begin{align*}
5 - 4 + 3 &= 2 \cdot 2 = 4 \\
\end{align*} \]
Claim \( 3n_f \geq n_e \)

1) add edge until \( G \) in connected
2) each face size in 3.
3) no parallel edges.

\[ 3n_f \leq 2n_e \]

\[ n_f \leq \frac{2}{3} n_e \]

\[ \frac{1}{3} n_e - n_e + n_f \geq 2 \]

\[ -\frac{1}{3} n_e + n_f \geq 2 \]

\[ n_f \geq a + \frac{1}{3} n_e \]

\[ 3n_f \geq n_e \]

Sweep line is \( O(n + I \log n) \) time

Wrong: we sorted the intersection!