CG 06
11/25/08

Bezier curves, de Casteljau Alg, Bernstein Poly

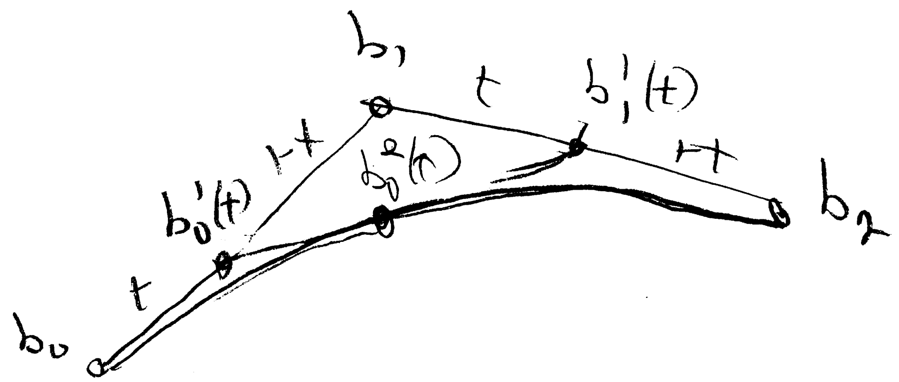
$b_0, b_1, b_2 \in R^2$ & $t \in R$

$$b_0^1(t) = (1-t)b_0 + t b_1$$

$$b_1^1(t) = (1-t)b_1 + t b_2$$

$$b_0^2(t) = (1-t)b_0^1(t) + t\, b_1^1(t)$$

picture



$$b_0^2(t) = (1-t)^2 b_0 \quad 2t(1-t)\, b_1 + t^2 b_0$$

$$(1-t)^2 + 2t(1-t) + t^2 = (1-t+t)^2 = 1$$

$$a^2 + 2ab + b^2 = (a+b)^2$$

de Castaljan Alg

$b_0 \cdots b_n \in R^d \quad t \in \mathbb{R}$

set $\quad b_i^r = (1-t) b_i^{r-1}(t) + t\, b_{i+1}^{r-1}(t) \qquad \begin{array}{l} r = 1, \cdots, n \\ i = 0, \cdots, n-r \end{array}$

b_i^r is poly curve of degree r

Claim de Casteljau commutes with affine maps

ie $x \to Ax+b$

$$\begin{tikzcd} \cdot \arrow[r, "Af"] \arrow[d, "Cast"'] & \cdot \arrow[d, "Cast"] \\ \cdot \arrow[r, "Af"'] & \cdot \end{tikzcd}$$

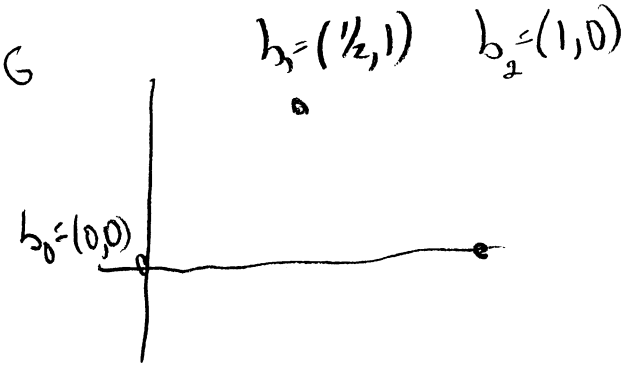
$$b_0^1(t) = (1-t)b_0 + tb_1$$

$$= (1-t)(Ab_0 + b) + t(Ab_1 + b)$$

$$= A((1-t)(b_0)) + (1-t)b + Atb_1 + tb)$$

$$A((1-t)b_0 + tb_1) + b$$

WLOG



Representing Curves
Implicit & Parametric

$t=0 \quad (1,-1)$

$t=1 \quad (3,0)$

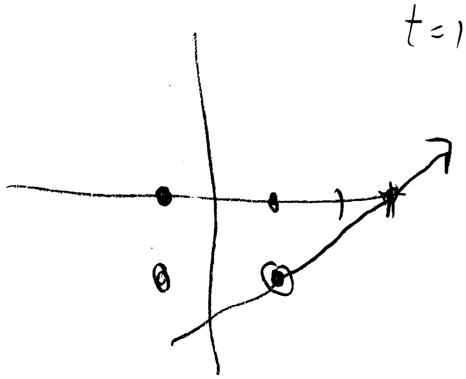
Parametric

$t=0 \quad t=1$

Line $\to \mathbb{R}^2$

$x = 2t+1$

$y = t-1$



Implicit

$x^2+y^2-1=0$

Parametric

$x = \cos\theta$

$y = \sin\theta$

Implicit $ax^2+bxy+cy^2+dx+ey+f=0$

Parabolas $y^2 = 4ax$

Ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Circles $a=b$

Hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Polynomial Curves of degree 1 & 2

Degree 1

$x(t) = a_1 t + a_0$

$y(t) = b_1 t + b_0$

$a_1, b_1 = 0$ then curve is a point

$b_1 x = a_1 b_1 t + a_0 b_1$

$a_1 y = a_1 b_1 t + a_1 b_0$

$$\boxed{b_1 x - a_0 b_1 = a_1 y + a_1 b_0}$$ linear implicit

Degree 2

$x(t) = F_1(t) = a_2 t^2 + a_1 t + a_0$

$y(t) = F_2(t) = b_2 t^2 + b_1 t + a_0$

$a_2 \neq 0$ or $b_2 \neq 0$ $\quad \rho = \sqrt{a_2^2 + b_2^2}$

$$R = \begin{pmatrix} b_2/\rho & -a_2/\rho \\ \frac{a_2}{\rho} & b_2/\rho \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix}$$ change of variables

$$x_1(t) = \left(\frac{b_2 a_2}{\rho} t^2 + \cdots - \frac{a_2 b_2 t^2}{\rho} + \cdots \right)$$

$$x_1(t) = a_1' t + a_0'$$
$$y_1(t) = b_2' t^2 + b_1' t + b_0' \quad b_2' \neq 0$$

$a_1' = 0$ degenerate

Goal eliminate b_1'

$$u = t + \frac{b_1'}{2b_2'}$$

Our form is

$$x(u) = a_1 u + a_0$$
$$y(u) = b_2 u^2 + b_0$$

a translation

$$x(u) = a u$$
$$y(u) = b u^2 \quad b > 0$$

$$\frac{b}{a^2}x^2 = bu^2 = y$$

<u>Thm</u> Parametric Quad are parabolas

<u>no</u> Ellipses or Hyperbolas

Computing Bezier by subdivision

fix $b_0 \cdots b_n$

$$
\begin{array}{ccccc}
b_0 & & & & \\
\vdots & b_0^1 & & & \\
\vdots & \vdots & b_0^2 & & \\
\vdots & \vdots & \vdots & \ddots & \\
b_n & b_{n-1}^1 & b_{n-2}^2 & \cdots & b_0^n
\end{array}
\qquad b_i^j \in \mathbb{R}^d
$$

two control poly

$$(b_0\ b_0^1 \cdots b_0^n)\quad (b_0^n\ b_{n-1}^{n-1} \cdots b_n^0)$$

<u>Claim</u> $\underset{0\le s\le 1}{(b_0 \cdots b_n)}\ \underset{0\le s\le t}{(b_0^1 \cdots b_0^n)}\ \underset{t\le s\le 1}{(b_0^n \cdots b_n^0)}$

all give the same curve.

<u>Menelao's Thm</u>

$$b[0,t] = (1-t)\,b_0 + t\,b_1$$

$$b[s,0] = (1-s)\,b_0 + s\,b_1$$

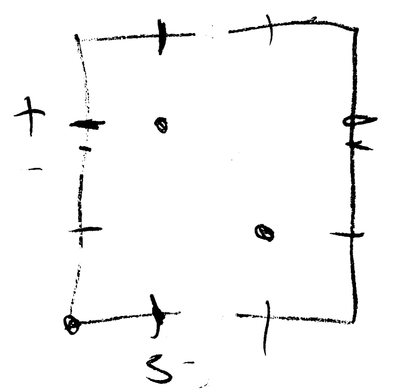
$$b[1,t] = (1-t)\,b_1 + t\,b_2$$

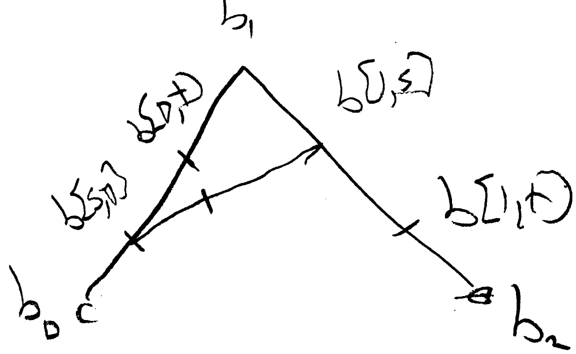
$$b[s,1] = (1-s)\,b_1 + s\,b_2$$

B

$$b[s,t] = (1-t)\,b[s,0] + t\,b[s,1]$$

$$b[t,s) = (1-s)\,b[0,t] + s\,b[1,t]$$

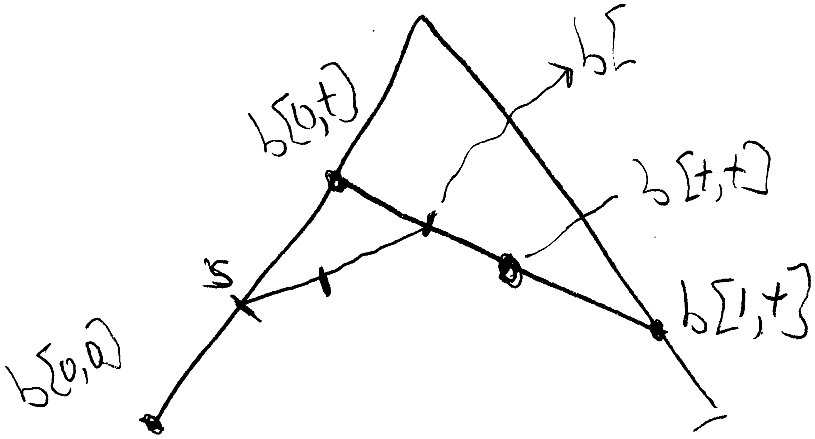




<u>Thm</u> $b[s,t] = b[t,s]$

<u>pf</u> by expansion

C



$b[0,0]$

b'
$\shortparallel$
$b[0,t] \quad b[0,s]$

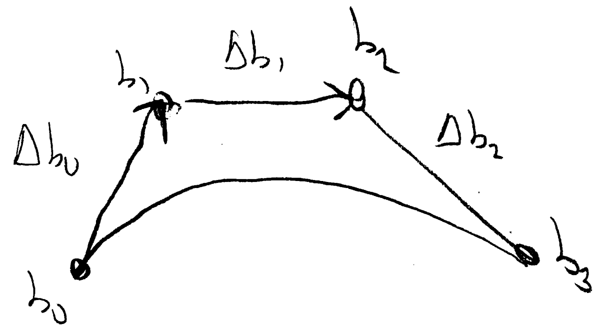
$b[t,t] \quad b[s,t] \quad b[s,s]$
$\shortparallel$
$b[t,s]$

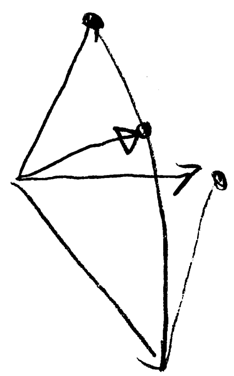
$b' = \frac{(1-s)}{t} b[0,0] + \frac{s}{t} [b[0,t]$

$= b[0,s]$

Fact $\alpha b[r, t_2] + \beta b[s, t_2] = b[\alpha r + \beta s, t_2] \quad \alpha + \beta = 1$

Derivative of a Bezier Curve





vector curve $(\Delta b_0, \Delta b_1, \Delta b_2)$

$\Delta b(t)$

$$\frac{\partial b(0)}{\partial t} = d \Delta b_0$$

d = degree