2D-LP Continued

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<u>Input</u>: 1) halfspaces $h_1 \cdots h_n$

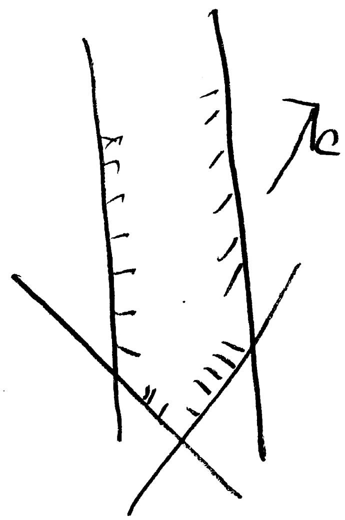
2) vector $c \in \mathbb{R}^2$

<u>Def</u> $F = \bigcap h_i$

<u>Output</u>: $\max_{x \in F} \{ x^T c \}$

<u>Note</u> X maybe unbounded

e.g.



<u>Assume</u>: 1) X is bounded

2) Halfspaces m_1 & m_2 st $F \subseteq m_1 \cap m_2$

3) c not normal to any h_i

Procedure 2D-LP$(m_1, m_2, h_1, \ldots, h_n, c)$

1) Set $v_0 =$ 2D-LP(m_1, m_2, c)
2) Randomly order $h_1 \ldots h_n$
3) For $i = 1$ to n,

a) If $v_{i-1} \in h_i$ then set $v_i = v_{i-1}$

Else Construct 1D-LP on $L = CH(h_i)$

$h_1' = L \cap h_1$

$\vdots$

$h_{i-1}' = L \cap h_{i-1}$

$c' = \text{proj}(c, L)$

Set $v_i =$ 1D-LP$(h_1', \ldots, h_i', c')$

b) If v_i unbded report "error"

v_i undef report "no solution"

else continue

<u>Correctness</u> $V_i = $ Correct-2D-LP$(m_1, \dots h_i, c)$ & unique

Induction on i

V_0 is OK by definition

Assume V_{i-1} is correct

May assume then V_{i-1} is defined.

1) If $V_{i-1} \in h_i$ then $V_{i-1} \in h_1 \cap \dots \cap h_i$

max $x^T c$

2) If $V_{i-1} \notin h_i$

<u>Claim</u> $V_i \in \partial h_i$

By uniqueness of V_{i-1} $\quad c^T V_i < c^T V_{i-1}$

Otherwise V_i works for $i-1$

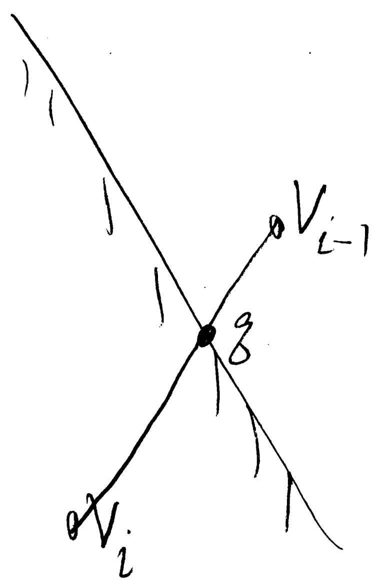
Suppose $V_i \notin \partial h_i$

Consider seg $S = [V_i, V_{i-1}]$

Let $q = S \cap \partial h_i$

then $S \subseteq F_i$ & $c^T q > c^T V_i$

contra!



Timing

Claim 2D-LP is $O(n)$ expected

Use backwards analysis

Suppose we remove random constrain h_j from $\{h_j, \ldots, h_i\}$

Def h_j is critical if solution changes

Charging Rule

$$\text{Charge}(h_j) = \begin{cases} K & \text{if } h_j \text{ not critical} \\ K \cdot i & \text{o.w.} \end{cases}$$

Note at most 2 critical constraints

Note 3 $\Rightarrow$ none

Expected cost $E_i \leq \dfrac{2(K \cdot i) + (i-2)K}{i} = \dfrac{3K \cdot i - 2K}{i} \leq 3K$

$E = \sum E_i \leq 3K \cdot n = O(n)$

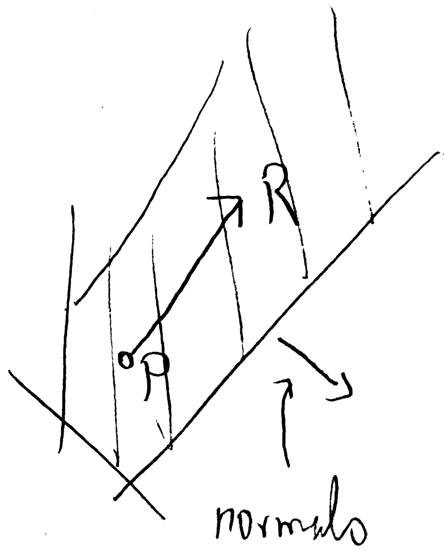
Unbounded Case of Finding bounding constraints

WLOG $\vec{c}$ is vertical

<u>Claim</u> Unbounded iff $\exists$ ray $R = \{P + \lambda\vec{d} : \lambda \geq 0\}$

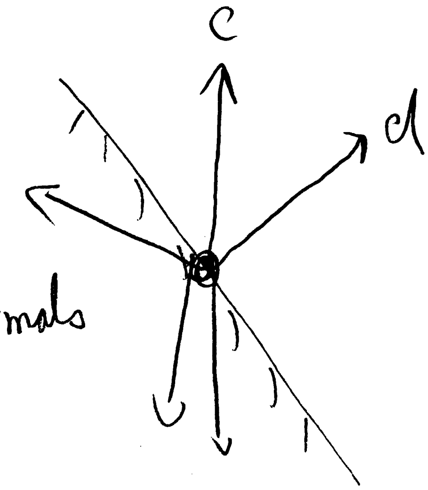
st 1) $R \subseteq$ feasible region

2) $c^T d > 0$



Let $n_1, \ldots, n_n$ be the normals.

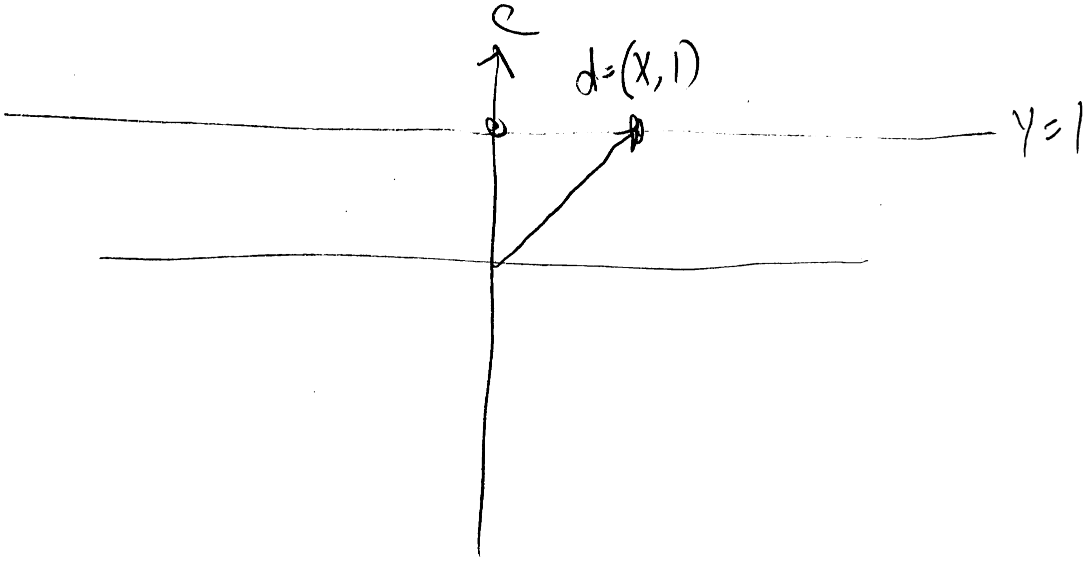
Consider



2D-LP: $d^T c > 0$

$d^T n_i \leq 0$

1D-LP Formulation



$\exists x$ st $n_x^{(i)} \cdot x + n_y^{(i)} \leq 0 \quad \forall i \equiv$ Bded-LP

Alg 1) Run Bded-LP

2) If Bded-LP feasible

a) more than one solution report "unbded".

b) unique d. check if LP feasible

Else Find h_i, h_j making Bded-LP infeasible

Set $m_1 = h_i$ & $m_2 = h_j$

Solve 2D-LP$(m_1, m_2, h_1, \dots, h_n, c)$