Springs and Graph Laplacian's

Input: Graph of Springs (Mattress)
\( G = (V, E, K) \)

\( K_{ij} \) = spring constant for edge \( E_{ij} \)

Consider only vertical displacements

\( u_i \) = displacement of \( V_i \)

God: Find solutions to Newton: \( f = ma \)
Find forces for displacement \( u = (u_1, \ldots, u_n) \).

Set force on \( u_i \) by spring \( E_{ij} \)

\[-(u_i - u_j) K_{ij}\]

ie linear spring model.
\[ L = L(G) \]

**Force vector** \(-LU\)

Let \( m_i \) = mass of \( V_i \)

\[ M = \begin{pmatrix} m_1 & 0 \\ 0 & m_n \end{pmatrix} \]

\( m_i > 0 \)

View \( u \) as a function of time \( u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix} \)

**Acceleration**: For \( u_i(t) = \frac{d^2 u_i}{dt^2} \)

\[ \frac{du}{dt} = \begin{pmatrix} \frac{du_1}{dt} \\ \vdots \\ \frac{du_n}{dt} \end{pmatrix} \quad \therefore \quad a_i = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \frac{d^2 u}{dt^2} \]

**Newton**: \(-LU = M(\frac{d^2 u}{dt^2})\)
Two ways to solve:

1) Solve for \( u \) given initial conditions 
   say \( u(0) & u'(0) \)

2) Find steady state solutions

Do 2) using Guess and check.

Recall: \( e^{iwt} = \cos \omega t + i \sin \omega t \)

\[
\frac{de^{iwt}}{dt} = i\omega e^{iwt} \equiv -\omega \sin \omega t + i\omega \cos \omega t
\]

\[
= i\omega (\cos \omega t + i \sin \omega t)
\]

\[
\frac{d^2 e^{iwt}}{dt^2} = \frac{d}{dt} \left( i\omega e^{iwt} \right) = i^2 \omega^2 e^{iwt} = -\omega^2 e^{iwt}
\]

\[
\frac{d\cos x}{dx} = -\sin x
\]

\[
\frac{d\sin x}{dx} = \cos x
\]
Guess: $U = e^{i\omega t} x$ some vector $x$.

$$\frac{d^2 U}{dt^2} = -W^2 e^{i\omega t} x$$

Check:

$$M(-W^2 e^{i\omega t} x) = -L(e^{i\omega t} x)$$

If $W^2 M x = L x$ set $\lambda = W^2$

If $L x = \lambda M x$ is $(\lambda, x)$ a generalized eigen-pair?

Claim: Eigenvalues of $L x = \lambda M x$ are real non-negative.

Change of variables $Y = M^{-\frac{1}{2}} x$ or $x = M^{-\frac{1}{2}} Y$

$$L M^{-\frac{1}{2}} Y = \lambda M M^{-\frac{1}{2}} Y$$

$$M^{-\frac{1}{2}} L M^{-\frac{1}{2}} Y = \lambda M^{-\frac{1}{2}} M M^{-\frac{1}{2}} Y$$

$k'$ is positive semi-definite
We have found a space of dimension solutions.

\[ W_i = \sqrt{\lambda_i} \quad \text{then} \quad X_i \quad \text{the eigenvector} \]

\[ u(t) = \alpha_i e^{i\omega t} X_i + \ldots + \alpha_n e^{i\omega t} X_n \]

is a solution.

Consider setting mass = the weight of degree

Eigenpairs \( L X = \lambda D X \) \((\lambda)\) \( L = D - A \)

Change of variable \( Y = D X \)

\( (\lambda) \text{iff } (D - A) D Y = \lambda Y \)

\( (I - AD)^{-1} Y = \lambda Y \quad M = AD^{-1} \quad \text{transition matrix} \)

for a random walk.

\[ Y - MY = \lambda Y \]

\[ Y - MY = (\lambda - 1) Y \quad \text{iff} \quad MY = (1 - \lambda) Y \]

If \( 0 = \lambda_1 < \lambda_2 \ldots \ldots \leq \lambda_n \) eigenvalues of \((\lambda)\)

then \( 0 \leq 1 - \lambda_n \ldots \ldots 1 - \lambda_{n-1} \), 0 eigen of \( MX = \lambda X \).
Differential Eq

\[
\frac{dx}{dt} = Ax \\
A^T = A
\]

\[
\lambda_i \leq \lambda_2 \leq \cdots \leq \lambda_n
\]

\[
x_i \quad = \quad x_n
\]

Guess & check \[ c e^{\lambda_i t} x_i = u \]

\[
\frac{du}{dt} = c \lambda_i e^{\lambda_i t} x_i
\]

\[
Au = A c e^{\lambda_i t} x_i = c e^{\lambda_i t} A x_i
\]

\[
= c e^{\lambda_i t} \lambda_i x_i
\]
\[ U(t) = c_1 e^{\lambda_1 t} x_1 + \ldots + c_n e^{\lambda_n t} x_n \]

\[ U(t) \text{ works!} \]

**Def.**
\[ e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots \]

\[ (e^{A_s})(e^{A_t}) = e^{A(s+t)} \quad (e^{A_t})(e^{-A_t}) = I \]

\[ \frac{d e^{A_t}}{dt} = A e^{A_t} \]

**Suppose** \[ A = S \Lambda S^{-1} \]

\[ e^{A_t} = I + S \Lambda S^{-1} + S \Lambda^2 S^{-1} t^2 + \ldots \]

\[ = S \left( I + \Lambda t + \ldots \right) S^{-1} = S e^{\Lambda t} S^{-1} \]
Solve \( \frac{du}{dt} = Au \) initial condition \( u(0) \)

Claim \( e^{At}u(0) \) works

\[
\frac{d}{dt} e^{At}u(0) = Ae^{At}u(0)
\]

Then \( A^T = A \) \( B^T = B \) then \( \text{tr}(e^{A+B}) = \text{tr}(e^A e^B) \)
GOLDEN-THOMPSON INEQUALITY

For $n \times n$ complex matrices, the matrix exponential is defined by Taylor series as

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$  

For commuting matrices $A$ and $B$ we see that $e^{A+B} = e^A e^B$ by multiplying the Taylor series. This identity is not true for general non-commuting matrices. In fact, it always fails if $A$ and $B$ do not commute, see [2].

**Theorem 1** (Golden-Thompson Inequality). For arbitrary self-adjoint matrices $A$ and $B$, one has

$$\text{tr}(e^{A+B}) \leq \text{tr}(e^A e^B).$$

For a survey of Golden-Thompson and other trace inequalities, see [2]. In the present note, we give a proof of Golden-Thompson inequality following [1] Theorem 9.3.7.


2. A version of Golden-Thompson inequality for three matrices fails:

$$\text{tr}(e^{A+B+C}) \not\leq \text{tr}(e^A e^B e^C).$$

The proof of Golden-Thompson inequality is based on Lie Product Formula:

**Theorem 2** (Lie Product Formula). For arbitrary matrices $A$ and $B$, we have

$$e^{A+B} = \lim_{N \to \infty} (e^{A/N} e^{B/N})^N.$$  

**Proof.** We first compare

$$X_N = e^{(A+B)/N} \quad \text{and} \quad Y_N = e^{A/N} e^{B/N}.$$  

As $N \to \infty$, Taylor’s expansion gives

$$X_N = 1 + \frac{A + B}{N} + O(N^{-2}),$$

$$Y_N = \left[1 + \frac{A}{N} + O(N^{-2})\right] \left[1 + \frac{B}{N} + O(N^{-2})\right]$$

$$= 1 + \frac{A}{N} + \frac{B}{N} + O(N^{-2}).$$

This shows that

$$(1) \quad X_N - Y_N = O(N^{-2}).$$
For a proof, see [1] Theorem 2.3.6.

Proposition 3 follows from Weyl’s Majorant Theorem for the function $f(x) = x^m$:

$$|\text{tr}(X^m)| = \left| \sum_{i=1}^{n} \lambda_i^m \right| \leq \sum_{i=1}^{n} |\lambda_i|^m \leq \sum_{i=1}^{n} s_i^m = \text{tr}(|X|^m).$$

\textit{Proof of Golden-Thompson Inequality.} Fix a natural number $N$ and consider

$$X = e^{A/2N}, \quad X = e^{B/2N}.$$  

To prove Golden-Thompson Inequality, it suffices to show that

\begin{equation}
\text{tr}((XY)^{2N}) \leq \text{tr}(X^{2N}Y^{2N}). \tag{2}
\end{equation}

Indeed, if (2) holds then, taking limit as $N \to \infty$ we see that the left hand side of (2) converges to $\text{tr}(e^{A+B})$ by Lie Product Formula, while the right hand side equals $\text{tr}(e^{A}e^{B})$.

To prove (2), we use Proposition 3 and note that $|XY|^2 = (XY)^*(XY) = YX^2Y$. We thus have

$$\text{tr}(XY)^{2N} \leq \text{tr}(YX^2Y)^{2N-1} = \text{tr}(X^2Y^2)^{2N-1},$$

where the last equality follows from the trace property $\text{tr}(UV) = \text{tr}(VU)$.

Continuing this procedure for $X^2$ and $Y^2$, we obtain

$$\text{tr}(X^2Y^2)^{2N-1} \leq \text{tr}(X^4Y^4)^{2N-2}.$$ 

After $N$ steps, we arrive at the bound (2). This proves Golden-Thompson Inequality. \qed

\textbf{References}
