Graph $G = (V, E, w)$ (possibly directed)

$w : E \rightarrow \mathbb{R}^+$

$w_i = w(v_i) = \sum_{(i, j) \in E} w_{ij} \quad P_{ij} = \frac{w_{ij}}{w_i}$

Random walk on $G$

Suppose, at a given time, we are at $v_i \in V$.

We move to $v_j$ with probability $P_{ij}$

E.g., $V$ = all permutations of a deck of cards

$P_{ij} = \text{prob of going from perm}_i \to \text{perm}_j$ in one shuffle.

? Why do professional players play from a deck after 5 shuffles?
Important Parameters

Access time or Hitting time

\[ H_{ij} = \text{Expected time to visit } j \text{ starting at } i \]

Commute Time

\[ K(i,j) = H(i,j) + H(i,i) \]

Cover Time

Expected time to visit all nodes
max over all starting nodes

Mixing Rate (to do)
Random walks - the Symmetric Case

Do a random walk on a network of conductors.

Input: \( G = (V, E, C) \) \( C_{ij} = C_{ji} \) \( a, b \in V \)

Consider a random walk starting at \( X \) and ending at \( b \).

Define \( h_x = \text{prob we visit } a \text{ before visiting } b \).

\( a \neq b \) starting from \( x \).

\[ 1 \quad 2 \quad 3 \quad 4 \quad 5 \]

\[ \begin{array}{c}
\text{EG} \\
\circ - \circ - \circ - \circ - \circ \\
a \quad x \quad b
\end{array} \]

\( h_a = 1 \quad h_b = 0 \)

\( h_a \text{?} \quad h_a > \frac{1}{2} \text{ why?} \)
\[ h_a = 1 \land h_b = 0 \]

Suppose \( X \neq a, b \)

Claim \( h_X = \sum \frac{P_{xy}}{P_{xx}} h_y \)

\( P_{xy} \geq 0 \quad \sum \frac{P_{xy}}{P_{xx}} = 1 \)

\( h_X \) is a convex combination of its neighbors!

\( h \) is harmonic with boundary \( a, b \)

Let's solve the electrical prob

\( V_a = 1 \) \& \( V_b = 0 \) and \( X \neq a, b \) float.

\( x \neq a, b \quad V_X = \sum \frac{C_{xy}}{C_x} V_y \) but \( \frac{C_{xy}}{C_x} = P_{xy} \)

\[ \Rightarrow h = V \]

Thm \( V_a = 1 \) \& \( V_b = 0 \) then \( V_X = \text{prob visit } a \text{ before } b \).
What does it mean (in random walks) if we set $V_{a_1} = V_{a_2} = 1$ & $V_{b_1} = V_{b_2} = 0$? $X \neq a_1, a_2, b_1, b_2$ float?
Interpretation of Current

Assume $G = (V, E, C)$ a) be V

Consider 1 unit of current flow from a to b,

Say i

What does $i_{xy}$ correspond to in random walks?

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**Thm** $i_{xy} = \text{Expected net } \# \text{ of traversals of } E_{xy}$

in random walk from a to b.

pf* Slides 7, 8, 8A
Let's start with:

\[ U_x = \text{Expected number of visits to } X \text{ before reaching } b \text{ starting at } a. \]

\[ U_b = 0 \quad x \neq b \]

\[ U_x = \sum_y U_y P_{yx} \quad \text{note } \sum_y P_{yx} \neq 1 \]

Recall \( C_x = \sum_y C_{xy} \)

\[ \text{note } \quad C_x P_{xy} = C_x \left( \frac{C_{xy}}{C_x} \right) = C_{xy} = C_{yx} = C_y \left( \frac{C_{yx}}{C_y} \right) = C_y P_{yx} \]

\[ U_x = \sum_y U_y \frac{C_y P_{yx}}{C_y} = \sum_y U_y \left( \frac{P_{xy} C_x}{C_y} \right) \]

\[ \frac{U_x}{C_x} = \sum_y P_{xy} \left( \frac{U_y}{C_y} \right) \]
Let $V_x = \left(\frac{U_x}{C_x}\right)$ then

$$V_x = \sum_Y P_{xy} V_y \quad V_y \text{ is harmonic}$$

What is the boundary! $V_b = 0$

Suppose we knew $U_a$, $V_a = \frac{U_a}{C_a}$

$V$ is a voltage where $V_b = 0$ & $V_a = \frac{U_a}{C_a}$

Let $J_{xy}$ be its current

$$J_{xy} = (V_x - V_y) C_{xy} = \left(\frac{U_x}{C_x} - \frac{U_y}{C_y}\right) C_{xy}$$

$$= U_x \left(\frac{C_{xy}}{C_x}\right) - U_y \left(\frac{C_{yx}}{C_y}\right) = U_x P_{xy} - U_y P_{yx}$$
$$U_x P_{xy} = \text{expected \# of traversals from } X \text{ to } Y$$
$$U_y P_{yx} = " Y \text{ to } X$$

$$i_{xy} = \text{expected \# net } xy \text{ traversals}$$

What is net current flow from \( a \) to \( b \)?

ie \( \sum Y i_{xy} \)

This must be 1

This proves Thm
How to compute hitting time

**Def:** $h(x, b)$ is expected time to reach $b$ from $x$

$h_x = h(x, b) \quad b$ fixed

Let's write a recurrence

$h_b = 0 \quad x \neq b$

\[ h_x = 1 + \sum_y h_y p_{xy} \quad \text{(*)} \]

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How do we solve (\*),?

Let's think of $h_x$ as a voltage $V_x$

$V_b = 0 \quad V_x = 1 + \sum_y \frac{C_{xy}}{C_x} V_y$
\[ C_x V_x = C_x + \sum_y C_{xy} V_y \]

\[ C_x V_x - \sum_y C_{xy} V_y = C_x \]

Graph Laplacian \hspace{1cm} \text{residual current}

\[
L V = \begin{pmatrix} C_1 \\ \vdots \\ C_{n-1} \\ 0 \end{pmatrix} \quad C = \sum C_i \quad b = V_n
\]

by conservation of flow

\[ \delta = C_n - C \]

\underline{Alg: for hitting time}

Solve \( L V = \begin{pmatrix} C_1 \\ \vdots \\ C_{n-1} \\ C_n - C \end{pmatrix} \)

\[ V_n = 0 \]

\[ \text{return } V_x \]
What about commute time?

$a = V_i$ & $b = V_n$

**Solution 1**

\[
LV^b = \begin{pmatrix} c_n \end{pmatrix} \\
\begin{pmatrix} \vdots \\ c_{n-1} \end{pmatrix} \\
\begin{pmatrix} c_{n-1} \\ c_{n-2} \\ \vdots \\ c_0 \end{pmatrix} \\
\begin{pmatrix} c_0 \\ 0 \\ \vdots \\ -c \end{pmatrix}
\]

\[
LV^a = \begin{pmatrix} c_i - c \end{pmatrix} \\
\begin{pmatrix} \vdots \\ c_{n-1} \end{pmatrix} \\
\begin{pmatrix} c_{n-1} \\ c_{n-2} \\ \vdots \\ c_0 \end{pmatrix} \\
\begin{pmatrix} c_0 \\ 0 \\ \vdots \\ -c \end{pmatrix}
\]

\[
h(1,n) = V_j^b - V_n^b
\]

\[
h(n,1) = V_n^a - V_1^a
\]

\[
V = V^b - V^a
\]

\[
c(1,n) = (V_j^b - V_n^b)_1 - (V_j^b - V_n^b)_n = V_j - V_n
\]

**Solution 2**

\[
L(V_j^b - V_j^a) = LV_j^b - LV_j^a = \begin{pmatrix} c_i \\ \vdots \\ c_{n-1} \end{pmatrix} - \begin{pmatrix} c_i - c \\ \vdots \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} c_i \\ \vdots \\ c_{n-1} \end{pmatrix}
\]

\[
\begin{pmatrix} c_0 \\ 0 \\ \vdots \\ -c \end{pmatrix}
\]

\[
\begin{pmatrix} c_i \\ \vdots \\ c_{n-1} \end{pmatrix} - \begin{pmatrix} c_i - c \\ \vdots \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} c_i \\ \vdots \\ c_{n-1} \end{pmatrix}
\]
solve \[ LV = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \]

return \( C(V, -V_n) \) but \( (V_i - V_n) = R_{mn} \)

\[ \text{Thm} \quad C(a, b) = \mathcal{B} R_{ab} \cdot C \]