

Resistive Models of a Graph & Random Walks

Making a recommendation (NETFLIX competition)

A bipartite graph: nodes - Viewers | Movies

weighted edges - weight is ranking by viewer of the movie

A distance metric:

$$\text{score}(v, m) = 1 / \text{rank}$$

Other metrics:

1. Shortest path

$$\text{score}(v, m) = \min_{vPm} w(P)$$

Problem: a maniac who likes all movies

⇒ all movies will be recommended to me

2. min-cut / max-flow

$$\text{score}(v, m) = \max \text{ flow from } v \text{ to } m$$

Problem: popular movies only? shortest paths do not improve score.

3. random walks... we'll start with electrical networks

Gary claims resistive networks do about the right thing (≈ shortest path AND max-flow)

□ Resistive Networks

More on resistive networks: Posted paper by Dole and Snell

Ohm's law

R : resistance

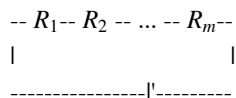
C : conductance ($C = 1/R$)

V : voltage

i : current

$$i = C \cdot V = V/R \tag{1}$$

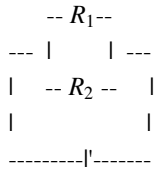
Resistors in series:



$$R = R_1 + \dots + R_m$$

$$C = 1/R = 1/(1/C_1 + \dots + 1/C_m)$$

Conductors in parallel:



$$C = C_1 + \dots + C_m$$

$$R = (1/R_1 + \dots + 1/R_m)^{-1}$$

Application: Siemens image segmentation: edges between pixels with weights = conductance

Effective Resistance / Conductance

Points a, b in a network

Voltage $V_{ab} = V_a - V_b$

Current $i_{ab} = i_a - i_b$

$$R_{ab} = V_{ab} / i_{ab}$$

e.g. use least effective resistance for recommendation

[HW] Show that R_{ab} is a metric, i.e.

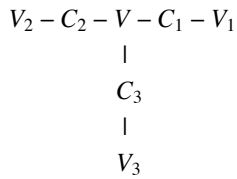
- (1) $R_{ab} \geq 0$
- (2) $R_{ab} = 0 \Rightarrow a = b$
- (3) $R_{ab} = R_{ba}$
- (4) $R_{ac} \leq R_{ab} + R_{bc}$, triangle inequality

Computing Effective Resistance

Before getting to computing the effective resistance, we need to introduce some other concepts.

An example

[Fig. 1] Graph \mathcal{G}



What happens at V:

$$i_1 = c_1(V - V_1)$$

$$i_2 = c_2(V - V_2)$$

$$i_3 = c_3(V - V_3)$$

$$i_1 + i_2 + i_3 \leftarrow \text{Residual current at } V$$

Suppose $i_1 + i_2 + i_3 = 0$. Then

$$c_1(V - V_1) + c_2(V - V_2) + c_3(V - V_3) = 0$$

$$(c_1 + c_2 + c_3)V = c_1 V_1 + c_2 V_2 + c_3 V_3$$

let $c = c_1 + c_2 + c_3$,

$$V = \frac{c_1}{c} V_1 + \frac{c_2}{c} V_2 + \frac{c_3}{c} V_3 \quad (2)$$

$\Rightarrow V$ is a convex combination of V_1, V_2, V_3 (useful)

Now Gary claims there is a natural matrix that occurs we should look at.

Consider graph $G = (V, E, w)$.

Adjacency matrix A :

$$A_{ij} = \begin{cases} w_{ij} & \text{if } (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Def Laplacian(G) = $L(G) = L$:

$$L_{ij} = \begin{cases} d(v_i) & \text{if } i = j \\ -w_{ij} & \text{if } (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

where

$$d(v_i) = \sum_{(v_i, v_j) \in E} w_{ij}$$

i.e.,

$$L = D - A, \quad D = \begin{pmatrix} d(v_1) & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & d(v_n) \end{pmatrix}$$

Note: for $V \equiv$ voltage setting of nodes of G ,

$$(LV)_i \equiv \text{residual current at } V_i$$

Example: consider graph \mathcal{G} of Fig.1. Its Laplacian is

$$L(G) = \begin{pmatrix} c_1 & 0 & 0 & -c_1 \\ 0 & c_2 & 0 & -c_2 \\ 0 & 0 & c_3 & -c_3 \\ -c_1 & -c_2 & -c_3 & c \end{pmatrix}$$

where $c = c_1 + c_2 + c_3$.

Then the residual current:

$$\begin{pmatrix} c_1 & 0 & 0 & -c_1 \\ 0 & c_2 & 0 & -c_2 \\ 0 & 0 & c_3 & -c_3 \\ -c_1 & -c_2 & -c_3 & c \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = \begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{pmatrix}$$

(Back to) Goal: compute effective resistance from V_1 to V_n

Method 1: Solve for V_2, \dots, V_{n-1}, i .

$$L \begin{pmatrix} 0 \\ V_2 \\ \dots \\ V_{n-1} \\ 1 \end{pmatrix} = \begin{pmatrix} i \\ 0 \\ \dots \\ 0 \\ -i \end{pmatrix}$$

L is singular -- each row adds up to a constant (0 here).

A magical solver will come back with numbers. Now how do we get R_{1n} ?

Recall from Ohm's law,

$$R = V / i$$

Since we have $V = 1, i = i$ (whatever came back from the solver)

$$R_{1n} = 1 / i$$

→ This is called a *boundary-valued problem*; V_1, V_n are the boundary points.

In other words, the problem is:

Compute (V_1, \dots, V_n) given $V_1 = 0, V_n = 1$.

Note: (V_1, \dots, V_n) is *harmonic*:

$\forall V_i \in \text{interior}, V_i \equiv \text{convex combination of neighbors}$

Maxwell's Principle

If $f : V \rightarrow R$ is harmonic, then the minimum and the maximum of f occur on the boundary.

Proof. V is interior \Rightarrow there exist neighbors V_i, V_j s.t.

$$V_i \leq V \leq V_j$$

Uniqueness Principle

If f and g are harmonic, with the same boundary conditions, then $f = g$.

Proof. $f - g : V \rightarrow R$ is harmonic with zeros on boundary. Hence $f - g = 0$ and $f = g$.

Method 2: Solve for V

$$LV = \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \\ 1 \end{pmatrix}$$

Then,

$$V_{1n} = V_1 - V_n$$

and since $i = 1$,

$$R_{1n} = V_1 - V_n$$

Raleigh's Principle

Consider another matrix, edge-vertex matrix, Γ

$$\Gamma^{E \times V} = \begin{array}{cccccc} & \square & v_1 & \dots & \dots & v_m \\ e_1 & \square & \square & \square & \square & \square \\ \dots & \square & \square & \square & \square & \square \\ e_n & \square & \square & \square & \square & \square \end{array}$$

Orient each edge

e.g.

$$\begin{array}{c} V_2 \leftarrow e_2 - V - e_1 \rightarrow V_1 \\ \uparrow \\ e_3 \\ | \\ V_3 \end{array}$$

gives the matrix

$$\begin{array}{ccccc} \square & v_1 & v_2 & v_3 & v_4 \\ e_1 & -1 & 0 & 0 & 1 \\ e_2 & 0 & -1 & 0 & 1 \\ e_3 & 0 & 0 & 1 & -1 \end{array}$$

Now let

$$C = \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & c_m \end{pmatrix}, \quad V = \begin{pmatrix} V_1 \\ \dots \\ \dots \\ V_m \end{pmatrix}$$

Then,

$$\Gamma V = \text{voltage change on each edge} \quad (3)$$

$$C \Gamma V = \text{current on each edge, by Ohm's law} \quad (4)$$

$$\Gamma^T C \Gamma V = \text{residual current at each vertex} \quad (5)$$

e.g. on the example above with $c_i = 1 \forall i$,

$$\Gamma^T C \Gamma V = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} V_4 - V_1 \\ V_4 - V_2 \\ V_3 - V_4 \end{pmatrix} = \begin{pmatrix} V_1 - V_4 \\ V_2 - V_4 \\ V_3 - V_4 \\ (V_4 - V_1) + (V_4 - V_2) + (V_4 - V_3) \\ = 3V_4 - V_1 - V_2 - V_3 \end{pmatrix}$$

$$\Gamma^T C \Gamma = L = D - A? \leftarrow \text{check that out yourself}$$

Anyway, that means L is a positive semi-definite matrix...!

Once we know what it is, Raleigh's principle would say that in the case of Netflix, adding more reviews decreases resistance.