15-859N Lecture 1, 9/11/2007

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Resistive Models of a Graph & Random Walks

Making a recommendation (NETFLIX competition)

A bipartite graph: nodes - Viewers | Movies

weighted edges - weight is ranking by viewer of the movie

A distance metric:

$$\text{score}(v, m) = 1/\text{rank}$$

Other metrics:

1. Shortest path

$$\text{score}(v, m) = \min_{\text{vPm}} w(P)$$

Problem: a maniac who likes all movies

⇒ all movies will be recommended to me

2. min-cut / max-flow

$$\text{score}(v, m) = \text{max flow from } v \text{ to } m$$

Problem: popular movies only? shortest paths do not improve score.

3. random walks... we'll start with electrical networks

Gary claims resistive networks do about the right thing (≈ shortest path AND max-flow)

Resistive Networks

More on resistive networks: Posted paper by Dole and Snell

Ohm's law

R : resistance
C : conductance ($C = 1/R$)
V : voltage
i : current

$$i = C \cdot V = V/R \qquad (1)$$

Resistors in series:

```
-- R1-- R2 -- ... -- Rm--
|                       |
----------------|'---------
```

$$R = R_1 + ... + R_m$$

$$C = 1/R = 1/(1/C_1 + ... + 1/C_m)$$

Conductors in parallel:

```
   -- R1--
---  |       | ---
|    -- R2 --    |
|                |
---------|'-------
```

$$C = C_1 + ... + C_m$$

$$R = (1 / R_1 + ... + 1 / R_m)^{-1}$$

Application: Siemens image segmentation: edges between pixels with weights = conductance

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Effective Resistance / Conductance

Points a, b in a network

Voltage $V_{ab} = V_a - V_b$

Current $i_{ab} = i_a - i_b$

$$R_{ab} = V_{ab} / i_{ab}$$

e.g. use least effective resistance for recommendation

[HW] Show that R_{ab} is a metric, i.e.

(1) $R_{ab} \geq 0$

(2) $R_{ab} = 0 \Rightarrow a = b$

(3) $R_{ab} = R_{ba}$

(4) $R_{ac} <= R_{ab} + R_{bc}$, triangle inequality

Computing Effective Resistance

Before getting to computing the effective resistance, we need to introduce some other concepts.

An example

[Fig.1] Graph $\mathcal{G}$

```
V2 - C2 - V - C1 - V1
          |
          C3
          |
          V3
```

What happens at V:

$i_1 = c_1(V - V_1)$

$i_2 = c_2(V - V_2)$

$i_3 = c_3(V - V_3)$

$i_1 + i_2 + i_3 \quad \leftarrow$ *Residual current* at V

Suppose $i_1 + i_2 + i_3 = 0$. Then

$$c_1(V - V_1) + c_2(V - V_2) + c_3(V - V_3) = 0$$

$$(c_1 + c_2 + c_3) V = c_1 V_1 + c_2 V_2 + c_3 V_3$$

let $c = c_1 + c_2 + c_3$,

$$V = \frac{c_1}{c} V_1 + \frac{c_2}{c} V_2 + \frac{c_3}{c} V_3 \tag{2}$$

$\Rightarrow V$ is a convex combination of V_1, V_2, V_3 (useful)

Now Gary claims there is a natural matrix that occurs we shoud look at.

Consider graph $G = (V, E, w)$.

Adjacency matrix A:

$$A_{\text{ij}} = \begin{cases} w_{\text{ij}} & \text{if } (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

<u>Def</u> Laplacian(G) = $L(G) = L$:

$$L_{\text{ij}} \begin{cases} d(V_i) & \text{if } i = j \\ -w_{\text{ij}} & \text{if } (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

where

$$d(v_i) = \sum w_{\text{ij}}, \; (v_i, v_j) \in E$$

i.e.,

$$L = D - A, \;\; D = \begin{pmatrix} d(v_1) & \ldots & 0 \\ 0 & \ldots & 0 \\ 0 & \ldots & d(v_n) \end{pmatrix}$$

<u>Note</u>: for $V \equiv$ voltage setting of nodes of G,

$$(\text{LV})_i \equiv \text{ residual current at } V_i$$

<u>Example</u>: consider graph $\mathcal{G}$ of Fig.1. Its Laplacian is

$$L(G) = \begin{pmatrix} c_1 & 0 & 0 & -c_1 \\ 0 & c_2 & 0 & -c_2 \\ 0 & 0 & c_3 & -c_3 \\ -c_1 & -c_2 & -c_3 & c \end{pmatrix}$$

where $c = c_1 + c_2 + c_3$.

Then the residual current:

$$\begin{pmatrix} c_1 & 0 & 0 & -c_1 \\ 0 & c_2 & 0 & -c_2 \\ 0 & 0 & c_3 & -c_3 \\ -c_1 & -c_2 & -c_3 & c \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = \begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{pmatrix}$$

(Back to) Goal: compute effective resistance from $V_1$to V_n

Method 1: Solve for $V_2, \ ..., V_{n-1}, i$.

$$L\begin{pmatrix} 0 \\ V_2 \\ ... \\ V_{n-1} \\ 1 \end{pmatrix} = \begin{pmatrix} i \\ 0 \\ ... \\ 0 \\ -i \end{pmatrix}$$

L is singular -- each row adds up to a constant (0 here).

A magical solver will come back with numbers. Now how do we get $R_{1\,n}$?

Recall from Ohm's law,

$$R = V / i$$

Since we have $V = 1$, $i = i$ (whatever came back from the solver)

$$R_{1\,n} = 1 / i$$

→ This is called a *boundary-valued problem*; V_1, V_n are the boundary points.

In other words, the problem is:

Compute $(V_1, \ ..., V_n)$ given $V_1 = 0, \ V_n = 1$.

Note: $(V_1, \ ..., V_n)$ is *harmonic:*

$\forall \ V_i \in$ interior, $V_i \equiv$ convex combination of neighbors

Maxwell's Principle

If $f : V \to R$ is harmonic, then the minimum and the maximum of f occur on the boundary.

Proof. V is interior ⇒ there exist neighbors V_i, V_j s.t.

$$V_i \le V \le V_j$$

Uniqueness Principle

If f and g are harmonic, with the same boundary conditions, then $f = g$.

Proof. $f - g : V \to R$ is harmonic with zeros on boundary. Hence $f - g = 0$ and $f = g$.

Method 2: Solve for V

$$\text{LV} = \begin{pmatrix} 1 \\ 0 \\ ... \\ 0 \\ 1 \end{pmatrix}$$

Then,

$$V_{1\,n} = V_1 - V_n$$

and since $i = 1$,

$$R_{1\,n} = V_1 - V_n$$

Raleigh's Principle

Consider another matrix, edge-vertex matrix, Γ

$$\Gamma^{|E|\times|V|} = \begin{array}{ccccc} \square & v_1 & \ldots & \ldots & v_m \\ e_1 & \square & \square & \square & \square \\ \ldots & \square & \square & \square & \square \\ e_n & \square & \square & \square & \square \end{array}$$

Orient each edge

e.g.

$$\begin{array}{c} V_2 \leftarrow e_2 - V - e_1 \rightarrow V_1 \\ \uparrow \\ e_3 \\ | \\ V_3 \end{array}$$

gives the matrix

$\square$	v_1	v_2	v_3	v_4
e_1	-1	0	0	1
e_2	0	-1	0	1
e_3	0	0	1	-1

Now let

$$C = \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & c_m \end{pmatrix}, \; V = \begin{pmatrix} V_1 \\ \ldots \\ \ldots \\ V_m \end{pmatrix}$$

Then,

$$\Gamma\, V \;=\; \text{voltage change on each edge} \tag{3}$$

$$C\,\Gamma\, V \;=\; \text{current on each edge, by Ohm's law} \tag{4}$$

$$\Gamma^T\, C\,\Gamma\, V \;=\; \text{residual current at each vertex} \tag{5}$$

e.g. on the example above with $c_i = 1\ \forall\ i$,

$$\Gamma^T\, C\,\Gamma\, V = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} V_4 - V_1 \\ V_4 - V_2 \\ V_3 - V_4 \end{pmatrix} = \begin{pmatrix} V_1 - V_4 \\ V_2 - V_4 \\ V_3 - V_4 \\ (V_4 - V_1) + (V_4 - V_2) + (V_4 - V_3) \\ = 3\,V_4 - V_1 - V_2 - V_3 \end{pmatrix}$$

$$\Gamma^T\, C\,\Gamma = L = D - A\,? \;\longleftarrow\; \text{check that out yourself}$$

Anyway, that means L is a positive semi-definite matrix...!

Once we know what it is, Raleigh's principle would say that in the case of Netflix, adding more reviews decreases resistance.