Applied Iterative Methods

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CHAPTER

3

Polynomial Acceleration

3.1 INTRODUCTION

In this chapter we describe a general procedure for accelerating the rates of convergence of basic iterative methods. This acceleration procedure, which we call *polynomial acceleration*, involves the formation of a new vector sequence from linear combinations of the iterates obtained from the basic method. As noted by Varga [1962], such a procedure is suggested by the theory of summability of sequences.

We define the general polynomial procedure, assuming only that the basic method is completely consistent (see Section 2.2). However, later when we consider Chebyshev and conjugate gradient polynomial methods, we generally assume that the basic method is also symmetrizable.

The polynomial procedure we present is but one of many approaches that may be used to accelerate the convergence of basic iterative methods. Some nonpolynomial acceleration methods are discussed briefly in Section 3.3.

3.2 POLYNOMIAL ACCELERATION OF BASIC ITERATIVE METHODS

Suppose the completely consistent basic method (2-1.2) is used to obtain approximations for the solution $\bar{u}$ of the nonsingular matrix problem $Au = b$.

Let the sequence of iterates generated by the basic method be given by $\{w^{(n)}\}$, i.e., given $w^{(0)}$, the sequence $\{w^{(n)}\}$ is formed by

$$w^{(n)} = Gw^{(n-1)} + k, \qquad n = 1, 2, \ldots. \tag{3-2.1}$$

From (2-2.6), the error vector $\tilde{\varepsilon}^{(n)} \equiv w^{(n)} - \bar{u}$ associated with the nth iterate of (3-2.1) satisfies

$$\tilde{\varepsilon}^{(n)} = G^n \tilde{\varepsilon}^{(0)}. \tag{3-2.2}$$

As a means to enhance the convergence of the $w^{(n)}$ iterates,† we consider a new vector sequence $\{u^{(n)}\}$ determined by the linear combination

$$u^{(n)} = \sum_{i=0}^{n} \alpha_{n,i} w^{(i)}, \qquad n = 0, 1, \ldots. \tag{3-2.3}$$

The only restriction we impose on the real numbers $\alpha_{n,i}$ is that

$$\sum_{i=0}^{n} \alpha_{n,i} = 1, \qquad n = 0, 1, \ldots. \tag{3-2.4}$$

This condition is imposed in order to ensure that $u^{(n)} = \bar{u}$ for all $n \geq 0$ whenever the initial guess vector $w^{(0)}$ is equal to the solution $\bar{u}$.

If we let $\varepsilon^{(n)} \equiv u^{(n)} - \bar{u}$ denote the error vector associated with the vectors $u^{(n)}$ of (3-2.3), we have from (3-2.3) and (3-2.4) that

$$\varepsilon^{(n)} = \sum_{i=0}^{n} \alpha_{n,i} w^{(i)} - \bar{u} = \sum_{i=0}^{n} \alpha_{n,i}(w^{(i)} - \bar{u}) = \sum_{i=0}^{n} \alpha_{n,i} \tilde{\varepsilon}^{(i)}.$$

Using (3-2.2), we then may express $\varepsilon^{(n)}$ in the form

$$\varepsilon^{(n)} = \left(\sum_{i=0}^{n} \alpha_{n,i} G^i \right) \tilde{\varepsilon}^{(0)}.$$

It follows from (3-2.3)–(3-2.4) that $\tilde{\varepsilon}^{(0)} = \varepsilon^{(0)}$. Thus, we may express $\varepsilon^{(n)}$ in the form

$$\varepsilon^{(n)} = Q_n(G)\varepsilon^{(0)}, \tag{3-2.5}$$

where $Q_n(G)$ is the *matrix polynomial* $Q_n(G) \equiv \alpha_{n,0} I + \alpha_{n,1} G + \cdots + \alpha_{n,n} G^n$. If $Q_n(x) \equiv \alpha_{n,0} + \alpha_{n,1} x + \cdots + \alpha_{n,n} x^n$ is the associated algebraic polynomial (see Section 1.3), then condition (3-2.4) requires that $Q_n(1) = 1$. This condition is the only restriction imposed thus far on $Q_n(x)$.

Because of the form (3-2.5) for the associated error vector, we call the combined procedure of (3-2.1) and (3-2.3) a *polynomial acceleration method*

† Since we have assumed only that the basic method (3-2.1) is completely consistent, convergence of the iterates $w^{(n)}$ to $\bar{u}$ is not guaranteed.

applied to the basic method (3-2.1). Varga [1962] calls this procedure a semi-iterative method with respect to the iterative method (3-2.1).

The high arithmetic cost and the large amount of storage required in using (3-2.3) to obtain $u^{(n)}$ make it necessary to seek alternative, less costly ways to compute $u^{(n)}$. We now show that a simpler computational form for $u^{(n)}$ is possible whenever the polynomials $Q_n(x)$ satisfy the recurrence relation

$$\begin{aligned} Q_0(x) &= 1, \\ Q_1(x) &= \gamma_1 x - \gamma_1 + 1, \\ Q_{n+1}(x) &= \rho_{n+1}(\gamma_{n+1}x + 1 - \gamma_{n+1})Q_n(x) + (1 - \rho_{n+1})Q_{n-1}(x) \quad \text{for } n \geq 1, \end{aligned} \tag{3-2.6}$$

where $\gamma_1, \rho_2, \gamma_2, \ldots$ are real numbers. Note that the $Q_n(x)$, defined by (3-2.6), satisfy $Q_n(1) = 1$ for all $n \geq 0$. We remark that the set of polynomial sequences $\{Q_n(x)\}$ satisfying (3-2.6) is large. For example, any properly normalized real orthogonal polynomial sequence is in such a set (see, e.g., Davis [1963]).

Theorem 3-2.1. Let the basic method (3-2.1) be completely consistent. If the polynomial sequence $\{Q_n(x)\}$ is given by (3-2.6), then the iterates $u^{(n)}$ of (3-2.3) may be obtained using the three-term relation

$$\begin{aligned} u^{(1)} &= \gamma_1(Gu^{(0)} + k) + (1 - \gamma_1)u^{(0)}, \\ u^{(n+1)} &= \rho_{n+1}\{\gamma_{n+1}(Gu^{(n)} + k) + (1 - \gamma_{n+1})u^{(n)}\} + (1 - \rho_{n+1})u^{(n-1)} \\ &\qquad \text{for} \quad n \geq 1. \end{aligned} \tag{3-2.7}$$

Conversely, any iterative procedure with iterates $u^{(n)}$ defined by (3-2.7) is equivalent to the polynomial procedures (3-2.1) and (3-2.3), with the polynomials $\{Q_n(x)\}$ given by (3-2.6).

Proof. Let the polynomials $\{Q_n(x)\}$ be given by (3-2.6) and let $\varepsilon^{(n)}$ be the error vector associated with the vector $u^{(n)}$ of (3-2.3). For $n \geq 1$, we have from (3-2.5) and (3-2.6) that

$$\varepsilon^{(n+1)} = \{\rho_{n+1}[\gamma_{n+1}G + (1 - \gamma_{n+1})I]Q_n(G) + (1 - \rho_{n+1})Q_{n-1}(G)\}\varepsilon^{(0)}, \tag{3-2.8}$$

and thus, again using (3-2.5), that

$$\varepsilon^{(n+1)} = \rho_{n+1}[\gamma_{n+1}G + (1 - \gamma_{n+1})I]\varepsilon^{(n)} + (1 - \rho_{n+1})\varepsilon^{(n-1)}. \tag{3-2.9}$$

By adding $\bar{u}$ to both sides of (3-2.9), we then obtain

$$\begin{aligned} u^{(n+1)} = \rho_{n+1}[\gamma_{n+1}G + (1 - \gamma_{n+1})I]u^{(n)} + (1 - \rho_{n+1})u^{(n-1)} \\ - \rho_{n+1}\gamma_{n+1}(G - I)\bar{u}. \end{aligned} \tag{3-2.10}$$

Now, using the fact that $\bar{u}$ also is a solution to the related system (2-2.2), we obtain the three-term form (3-2.7). The special case for $\varepsilon^{(1)}$ follows similarly. Conversely, let $\hat{\varepsilon}^{(n)} \equiv u^{(n)} - \bar{u}$ be the error vector associated with the vectors $u^{(n)}$ of (3-2.7). By reversing the above steps using $\hat{\varepsilon}^{(n+1)}$ in place of $\varepsilon^{(n+1)}$, we get $\hat{\varepsilon}^{(n+1)} = Q_{n+1}(G)\hat{\varepsilon}^{(0)}$ with $Q_{n+1}(x)$ defined by (3-2.6). From this, it follows that the iterative procedure (3-2.7) is equivalent to a polynomial acceleration method, with the polynomials given by (3-2.6). ■

As noted previously, there are many polynomial sequences $\{Q_n(x)\}$ that satisfy (3-2.6). In this book, we consider only those polynomial sequences that are associated with the Chebyshev and conjugate gradient acceleration methods. We discuss below the general basis on which the polynomials $\{Q_n(x)\}$ are chosen for these methods.

We now assume that the basic method (3-2.1) is also *symmetrizable* with a symmetrization matrix W. From (3-2.5), we have for any vector norm $\|\cdot\|_L$ that

$$\|\varepsilon^{(n)}\|_L = \|Q_n(G)\varepsilon^{(0)}\|_L. \tag{3-2.11}$$

For the conjugate gradient method, the polynomial sequence $\{Q_n(x)\}$ is chosen to minimize $\|Q_n(G)\varepsilon^{(0)}\|_L$, or equivalently $\|\varepsilon^{(n)}\|_L$ for a particular choice of the vector norm $\|\cdot\|_L$. The conjugate gradient method is discussed in Chapter 7.

Since WGW^{-1} is symmetric, it follows that the matrix $WQ_n(G)W^{-1}$ is also symmetric. Thus from (1-4.7) and (1-4.11), we have that

$$\|\varepsilon^{(n)}\|_W \leq \|Q_n(G)\|_W \|\varepsilon^{(0)}\|_W = \mathbf{S}(Q_n(G))\|\varepsilon^{(0)}\|_W. \tag{3-2.12}$$

For the Chebyshev acceleration procedure, the error norm $\|\varepsilon^{(n)}\|_W$ is made small by picking the polynomials $\{Q_n(x)\}$ such that the spectral radius $\mathbf{S}(Q_n(G))$ is small. More precisely, let $\{\mu_i\}_{i=1}^N$ be the set of eigenvalues for the $N \times N$ matrix G. Then $\{Q_n(\mu_i)\}_{i=1}^N$ is the set of eigenvalues for the matrix $Q_n(G)$. (See Section 1.3.) Thus we have

$$\mathbf{S}(Q_n(G)) = \max_{1 \leq i \leq N} |Q_n(\mu_i)|. \tag{3-2.13}$$

Since the complete eigenvalue spectrum of G is seldom known, it is more convenient to consider the *virtual spectral radius* of $Q_n(G)$ in place of $\mathbf{S}(Q_n(G))$. If $M(G)$ and $m(G)$ denote, respectively, the algebraically largest and smallest eigenvalues of G, then the virtual spectral radius of $Q_n(G)$ is defined by

$$\bar{\mathbf{S}}(Q_n(G)) \equiv \max_{m(G) \leq x \leq M(G)} |Q_n(x)|. \tag{3-2.14}$$

Since the set of eigenvalues $\{\mu_i\}_{i=1}^N$ is in the interval $[m(G), M(G)]$, we have that

$$\mathbf{S}(Q_n(G)) \leq \bar{\mathbf{S}}(Q_n(G)). \tag{3-2.15}$$

For the Chebyshev method, the polynomial sequence $\{Q_n(x)\}$ is chosen such that $\bar{\mathbf{S}}(Q_n(G))$ is minimized. The Chebyshev method is described in Chapter 4.

Analogous to the definitions given in Section 2.2, the *virtual average rate of convergence for a polynomial method* is defined by

$$\bar{R}_n(Q_n(G)) \equiv -(1/n)\log \bar{\mathbf{S}}(Q_n(G)), \tag{3-2.16}$$

and provided the limit exists, the *virtual asymptotic rate of convergence* is given by

$$\bar{R}_\infty(Q_n(G)) \equiv \lim_{n\to\infty} \bar{R}_n(Q_n(G)). \tag{3-2.17}$$

3.3 EXAMPLES OF NONPOLYNOMIAL ACCELERATION METHODS

In this section we describe briefly several alternatives to the polynomial approach for accelerating the convergence of basic iterative methods. The classes of general acceleration procedures we present are called acceleration by additive correction and acceleration by multiplicative correction.

Suppose the basic iterative process

$$u^{(n)} = Gu^{(n-1)} + k \tag{3-3.1}$$

is used to obtain approximations for the solution $\bar{u}$ of the matrix problem $Au = b$. It easily follows that the error vector $\varepsilon^{(n)} \equiv u^{(n)} - \bar{u}$ and the *residual vector* $r^{(n)} \equiv Au^{(n)} - b$ satisfy the residual equation

$$A\varepsilon^{(n)} = r^{(n)}. \tag{3-3.2}$$

If Eq. (3-3.2) can be solved for $\varepsilon^{(n)}$, then we immediately have the solution $\bar{u} = u^{(n)} - \varepsilon^{(n)}$. However, it is as difficult to solve (3-3.2) for $\varepsilon^{(n)}$ as it is to solve $Au = b$ for $\bar{u}$. On the other hand, it is not always necessary to determine $\varepsilon^{(n)}$ with great precision to improve the accuracy of $u^{(n)}$. Thus the basic method (3-3.1) can often be accelerated by using the following procedure:

(1) Do L iterations of the basic method (3-3.1), using $u^{(0)}$ as the initial guess.

(2) Compute $\tilde{\varepsilon}^{(L)}$, where $\tilde{\varepsilon}^{(L)}$ is some approximation to $\varepsilon^{(L)}$ of (3-3.2), and set $u^{(0)} = u^{(L)} - \tilde{\varepsilon}^{(L)}$. Then go to step (1) again.

There are many ways to obtain the approximation $\tilde{\varepsilon}^{(L)}$. For example, if A corresponds to the discretization with a mesh π_h of a continuous operator, then $\tilde{\varepsilon}^{(L)}$ may be taken to satisfy $\tilde{A}\tilde{\varepsilon}^{(L)} = \tilde{r}^{(L)}$, where $\tilde{A}$ corresponds to a discretization over a coarser mesh, say π_{2h}. Some methods that utilize this

general approach are "the synthetic method" (e.g., see Kopp [1963] and Gelbard and Hageman [1969]), "multilevel methods" (e.g., see Brandt [1977]), and "multigrid methods" (e.g., see Nicolaides [1975, 1977] and Hackbusch [1977]). Another approach is to obtain the approximation $\tilde{\varepsilon}^{(L)}$ using the method of weighted residuals. To do this, let $\tilde{\varepsilon}^{(L)}$ be written as

$$\tilde{\varepsilon}^{(L)} = \sum_{i=1}^{M} c_i \alpha_i, \tag{3-3.3}$$

where the α_i are some known vectors.† The unknown constants c_i are determined from the M equations

$$w_i^{\mathrm{T}} A \tilde{\varepsilon}^{(L)} = w_i^{\mathrm{T}} r^{(L)}, \qquad i = 1, \ldots, M, \tag{3-3.4}$$

where the w_i are known weighting vectors. See, for example, de la Vallee Poussin [1968] and Setturi and Aziz [1973].

The methods discussed above are called *additive correction acceleration methods*. However, *multiplicative correction methods* have also been used. Multiplicative correction methods attempt to improve the accuracy of $u^{(L)}$ by multiplying $u^{(L)}$ by some matrix E instead of adding a vector $\tilde{\varepsilon}^{(L)}$ as in step (2) above. Usually, E is a diagonal matrix whose diagonal entries are determined by some weighted residual or variational method. Descriptions of methods based on the multiplicative correction approach are given, for example, by Kellogg and Noderer [1960] (scaled iterations), Nakamura [1974] (coarse mesh rebalancing techniques), and Wachspress [1966] (coarse mesh variational techniques). Wachspress [1977] considers an acceleration procedure based on combined additive and multiplicative correction.

† Often, the elements of α_i are chosen to be either 0 or 1.

CHAPTER

4

Chebyshev Acceleration

4.1 INTRODUCTION

In this chapter we consider Chebyshev polynomial acceleration applied to basic iterative methods of the form

$$u^{(n)} = Gu^{(n-1)} + k, \qquad n = 1, 2, \ldots. \tag{4-1.1}$$

We assume throughout this chapter that the iterative method (4-1.1) *is symmetrizable.*

Recall from Chapter 3 that the error vector associated with a general polynomial acceleration procedure applied to (4-1.1) can be expressed as

$$\varepsilon^{(n)} = Q_n(G)\varepsilon^{(0)}, \tag{4-1.2}$$

where $Q_n(G) \equiv \alpha_{n,0}I + \alpha_{n,1}G + \cdots + \alpha_{n,n}G^n$ is a matrix polynomial subject only to the condition that $\sum_{i=0}^{n} \alpha_{n,i} = 1$. Using the algebraic polynomial $Q_n(x)$ associated with $Q_n(G)$, we defined

$$\bar{\mathbf{S}}(Q_n(G)) \equiv \max_{m(G) \le x \le M(G)} |Q_n(x)| \tag{4-1.3}$$

as the virtual spectral radius of the matrix $Q_n(G)$. As before, $M(G)$ and $m(G)$ denote, respectively, the algebraically largest and smallest eigenvalues of G. That particular polynomial method which is obtained by choosing the

polynomial sequence $\{Q_n(G)\}$ such that $\bar{\mathbf{S}}(Q_n(G))$, $n = 1, 2, \ldots$, is minimized is called the Chebyshev polynomial acceleration method.

In Section 4.2, we show that, indeed, the matrix polynomial $Q_n(G)$ that minimizes $\bar{\mathbf{S}}(Q_n(G))$ can be defined in terms of Chebyshev polynomials. The basic Chebyshev computational procedure is also derived in Section 4.2. It turns out that the proper application of Chebyshev acceleration requires the use of "iteration parameters" whose optimum values are functions of the extreme eigenvalues $M(G)$ and $m(G)$ of G. When optimum iteration parameters are used, we show that Chebyshev acceleration can significantly improve the convergence rate. For most practical applications, however, the optimum parameters will not be known a priori and must be approximated by some means. In Sections 4.3 and 4.4, we study the behavior of the Chebyshev method when iteration parameters which are not optimum are used. Computational algorithms which generate the necessary Chebyshev iteration parameters adaptively during the iteration process are presented in Chapters 5 and 6.

4.2 OPTIMAL CHEBYSHEV ACCELERATION

We first show that the matrix polynomial $Q_n(G)$ which minimizes $\bar{\mathbf{S}}(Q_n(G))$ is unique and can be defined in terms of Chebyshev polynomials. For any nonnegative integer n, the Chebyshev polynomial† of degree n in w may be defined by the recurrence relation

$$\begin{aligned} T_0(w) &= 1, \qquad T_1(w) = w, \\ T_{n+1}(w) &= 2wT_n(w) - T_{n-1}(w), \qquad n \geq 1. \end{aligned} \tag{4-2.1}$$

It can be shown by mathematical induction (see, e.g., Young [1971]) that the $T_n(w)$ may also be expressed by

$$\begin{aligned} T_n(w) &= \tfrac{1}{2}[(w + \sqrt{w^2 - 1})^n + (w + \sqrt{w^2 - 1})^{-n}] \\ &= \tfrac{1}{2}[(w - \sqrt{w^2 - 1})^n + (w - \sqrt{w^2 - 1})^{-n}] \\ &= \cosh(n \cosh^{-1} w) \qquad \text{when} \quad w > 1, \\ &= \cos(n \cos^{-1} w) \qquad \text{when} \quad -1 \leq w \leq 1. \end{aligned} \tag{4-2.2}$$

We note that $T_n(w)$ is an even function of w for n even and an odd function of w for n odd.

The fundamental properties of Chebyshev polynomials that we shall use are given in the following theorem.

† The Chebyshev polynomials utilized in this book are the so-called Chebyshev polynomials of the first kind.

Theorem 4-2.1. Let n be a fixed integer and let d be any fixed real number such that $d > 1$. If we let

$$H_n(w) = T_n(w)/T_n(d), \tag{4-2.3}$$

where $T_n(w)$ is the Chebyshev polynomial (4-2.1), then

$$H_n(d) = 1 \tag{4-2.4}$$

and

$$\max_{-1 \le w \le 1} |H_n(w)| = 1/T_n(d). \tag{4-2.5}$$

Moreover, if $Q(w)$ is any polynomial of degree n or less such that $Q(d) = 1$ and

$$\max_{-1 \le w \le 1} |Q(w)| \le \max_{-1 \le w \le 1} |H_n(w)|,$$

then

$$Q(w) = H_n(w). \tag{4-2.6}$$

Proof. See, for example, Young [1971] or Flanders and Shortly [1950]. ■

Returning now to the problem of minimizing $\bar{\mathbf{S}}(Q_n(G))$, we seek that polynomial $P_n(x)$ such that $P_n(1) = 1$ and such that

$$\max_{m(G) \le x \le M(G)} |P_n(x)| \le \max_{m(G) \le x \le M(G)} |Q_n(x)|, \tag{4-2.7}$$

where $Q_n(x)$ is any polynomial of degree n or less satisfying $Q_n(1) = 1$. The existence and definition of such a polynomial follows from Theorem 4-2.1. Specifically, let

$$w(x) \equiv (2x - M(G) - m(G))/(M(G) - m(G)) \tag{4-2.8}$$

be the linear transformation which maps the interval $m(G) \le x \le M(G)$ onto the interval $-1 \le w \le 1$ and let

$$H_n(w) \equiv T_n(w(x))/T_n(w(1)).$$

Since the basic method (4-1.1) is symmetrizable, it follows from Theorem 2-2.1 that $M(G) < 1$. Thus $w(1) > 1$. If we now define $P_n(x)$ as

$$P_n(x) \equiv T_n\left(\frac{2x - M(G) - m(G)}{M(G) - m(G)}\right) \Big/ T_n\left(\frac{2 - M(G) - m(G)}{M(G) - m(G)}\right), \tag{4-2.9}$$

then obviously

$$\max_{m(G) \le x \le M(G)} |P_n(x)| = \max_{-1 \le w \le 1} |H_n(w)|. \tag{4-2.10}$$

It now follows from Theorem 4-2.1 that the $P_n(x)$ of (4-2.9) is the unique polynomial satisfying (4-2.7). Thus we have

Theorem 4-2.2. Let $\mathcal{S}_n$ be the set of polynomials $\{Q_n(x)\}$ of degree n or less satisfying $Q_n(1) = 1$. Then, the polynomial $P_n(x)$ of (4-2.9) is the unique polynomial in the set $\mathcal{S}_n$ which satisfies

$$\max_{m(G) \le x \le M(G)} |P_n(x)| \le \max_{m(G) \le x \le M(G)} |Q_n(x)|$$

for any $Q_n(x) \in \mathcal{S}_n$.

We now consider computational and convergence aspects of polynomial acceleration using $P_n(x)$ when applied to the basic method (4-1.1). Using (4-2.1), it is easy to show that the polynomials $P_n(x)$ satisfy the recurrence relation

$$P_0(x) = 1, \qquad P_1(x) = \bar{\gamma}x - \bar{\gamma} + 1,$$
$$P_{n+1}(x) = \bar{\rho}_{n+1}(\bar{\gamma}x + 1 - \bar{\gamma})P_n(x) + (1 - \bar{\rho}_{n+1})P_{n-1}(x) \qquad \text{for} \quad n \ge 1, \tag{4-2.11}$$

where

$$\bar{\gamma} = 2/(2 - M(G) - m(G)), \tag{4-2.12}$$

$$\bar{\rho}_{n+1} = 2w(1)T_n(w(1))/T_{n+1}(w(1)), \tag{4-2.13}$$

and where $w(x)$ is defined by (4-2.8). It now follows from Theorem 3-2.1 that the iterates for the polynomial procedure based on $P_n(x)$ may be obtained by using the three-term recurrence relation

$$u^{(n+1)} = \bar{\rho}_{n+1}\{\bar{\gamma}(Gu^{(n)} + k) + (1 - \bar{\gamma})u^{(n)}\} + (1 - \bar{\rho}_{n+1})u^{(n-1)}. \tag{4-2.14}$$

We refer to the method defined by (4-2.14) with $\bar{\gamma}$ and $\bar{\rho}_{n+1}$ given by (4-2.12) and (4-2.13) as the *optimal Chebyshev acceleration procedure*. The term "optimal" is used to distinguish the method from other (nonoptimal) procedures in which estimates m_E and M_E are used for $m(G)$ and $M(G)$, respectively. Such procedures are described in Section 4.3.

Again making use of the Chebyshev polynomial recurrence relation (4-2.1), we can write the parameters $\bar{\rho}_{n+1}$ of (4-2.13) in the more computationally convenient form

$$\bar{\rho}_1 = 1, \qquad \bar{\rho}_2 = (1 - \tfrac{1}{2}\bar{\sigma}^2)^{-1},$$
$$\bar{\rho}_{n+1} = (1 - \tfrac{1}{4}\bar{\sigma}^2\bar{\rho}_n)^{-1}, \qquad n \ge 2, \tag{4-2.15}$$

where

$$\bar{\sigma} = 1/w(1) = (M(G) - m(G))/(2 - M(G) - m(G)). \tag{4-2.16}$$

It can also be shown (Varga, 1962) that

$$\lim_{n\to\infty} \bar{\rho}_n \equiv \bar{\rho}_\infty = 2/(1 + \sqrt{1 - \bar{\sigma}^2}). \tag{4-2.17}$$

We now examine the convergence rate of the optimal Chebyshev procedure. From (4-1.3), (4-2.5), and (4-2.10), the virtual spectral radius of $P_n(G)$ is

$$\bar{\mathbf{S}}(P_n(G)) = [T_n(w(1))]^{-1} = [T_n(1/\bar{\sigma})]^{-1}. \tag{4-2.18}$$

Using (4-2.2), and after a small amount of algebra, we can write $T_n(1/\bar{\sigma})$ in the form

$$T_n\left(\frac{1}{\bar{\sigma}}\right) = \frac{1 + \bar{r}^n}{2\bar{r}^{n/2}},$$

where

$$\bar{r} \equiv (1 - \sqrt{1 - \bar{\sigma}^2})/(1 + \sqrt{1 - \bar{\sigma}^2}). \tag{4-2.19}$$

Thus we obtain

$$\bar{\mathbf{S}}(P_n(G)) = 2\bar{r}^{n/2}/(1 + \bar{r}^n). \tag{4-2.20}$$

Thus from (3-2.16) and (4-2.20), the average virtual rate of convergence for the optimal Chebyshev method is

$$\bar{R}_n(P_n(G)) = -\frac{1}{2}\log \bar{r} - \frac{1}{n}\log\left(\frac{2}{1 + \bar{r}^n}\right). \tag{4-2.21}$$

It is easy to see from (4-2.21) that the asymptotic rate of convergence defined by (3-2.17) can be expressed in the form

$$\bar{R}_\infty(P_n(G)) = -\tfrac{1}{2}\log \bar{r}. \tag{4-2.22}$$

Note from (4-2.17) and (4-2.19) that $\bar{r} = \bar{\rho}_\infty - 1$. Thus we also have that

$$\lim_{n\to\infty} [\bar{\mathbf{S}}(P_n(G))]^{1/n} = (\bar{\rho}_\infty - 1)^{1/2} \quad \text{and that} \quad \bar{R}_\infty(P_n(G)) = -\tfrac{1}{2}\log(\bar{\rho}_\infty - 1).$$

From (4-2.21) and (4-2.22), it easily follows that $\bar{R}_n(P_n(G)) < \bar{R}_\infty(P_n(G))$ for all finite n. In fact, it can be shown that $\bar{R}_n(P_n(G))$ is an increasing function of n. We omit the proof. However, the data given in Table 4-2.1 show that many iterations often are required before the asymptotic convergence is achieved. In Table 4-2.1, we tabulate the values of the ratio

$$\frac{\bar{R}_n(P_n(G))}{\bar{R}_\infty(P_n(G))} = \frac{-\log(2\bar{r}^{n/2}/(1 + \bar{r}^n))}{-\log \bar{r}^{n/2}} = 1 + \frac{\log(2/(1 + \bar{r}^n))}{\log \bar{r}^{n/2}} \tag{4-2.23}$$

as a function of $\bar{r}$. Thus we see that if, after n iterations, $\bar{r}^n \doteq 0.1$, then the average virtual convergence rate for these n iterations is only about one-half

TABLE 4-2.1

Values of $\bar{R}_n(P_n(G))/\bar{R}_\infty(P_n(G))$ *as a Function of* $\bar{r}^n$

$\bar{r}^n$	$\bar{R}_n(P_n(G))/\bar{R}_\infty(P_n(G))$	$\bar{r}^n$	$\bar{R}_n(P_n(G))/\bar{R}_\infty(P_n(G))$
10^{-1}	0.481	10^{-6}	0.900
10^{-2}	0.703	10^{-12}	0.948
10^{-3}	0.800	10^{-24}	0.975
10^{-4}	0.850	0	1.000
10^{-5}	0.880		

of its value for later iterations when the asymptotic convergence rate is achieved.

We now compare the optimum Chebyshev acceleration procedure with the optimum extrapolated procedure defined in Section 2.2. For the optimum extrapolated method applied to the basic method (4-1.1), we have by (2-2.16) and (4-2.16) that $\mathbf{S}(G_{[\bar{\gamma}]}) = \bar{\sigma}$ and hence that

$$R_\infty(G_{[\bar{\gamma}]}) = -\log \bar{\sigma}. \tag{4-2.24}$$

For the optimal Chebyshev procedure applied to (4-1.1), we have from (4-2.22) that $\bar{R}_\infty(P_n(G)) = -\frac{1}{2}\log \bar{r}$, where $\bar{r}$ is given by (4-2.19). It is easy to show that

$$-\log \bar{\sigma} \sim 1 - \bar{\sigma}, \qquad \bar{\sigma} \to 1-$$

and that

$$-\tfrac{1}{2}\log \bar{r} \sim \sqrt{1-\bar{\sigma}^2} \sim \sqrt{2}\sqrt{1-\bar{\sigma}}, \qquad \bar{\sigma} \to 1-.$$

Combining this with (4-2.22) and (4-2.24), we have

$$\bar{R}_\infty(P_n(G)) \sim \sqrt{2}\sqrt{R_\infty(G_{[\bar{\gamma}]})}, \qquad \bar{\sigma} \to 1-. \tag{4-2.25}$$

Thus for $\bar{\sigma}$ close to unity, the optimum Chebyshev procedure is an order of magnitude faster than the optimum extrapolated procedure.

As an example, consider the case in which $M(G) = -m(G) = 0.99$. In this case we have $\bar{\sigma} = 0.99$ and $\bar{r} \doteq 0.753$. Thus each iteration of the optimum extrapolated method reduces the error by approximately a factor of 0.99, while each iteration of the optimum Chebyshev method reduces the error by approximately a factor of 0.868. The number of iterations needed to reduce the norm of the error vector by a factor of 10^{-6}, as compared with the norm of the initial error vector, would be approximately 1375 for the extrapolated method and 98 for the optimum Chebyshev method. The factor of improvement would be greater for a larger value of $\bar{\sigma}$.

4.3 CHEBYSHEV ACCELERATION WITH ESTIMATED EIGENVALUE BOUNDS

We now study the behavior of the Chebyshev acceleration process when estimates m_E and M_E are used for $m(G)$ and $M(G)$, respectively. When these estimates are used, the normalized Chebyshev polynomial of (4-2.9) is written as

$$P_{n,E}(x) \equiv T_n\left(\frac{2x - M_E - m_E}{M_E - m_E}\right) \Big/ T_n\left(\frac{2 - M_E - m_E}{M_E - m_E}\right) = \frac{T_n(w_E(x))}{T_n(w_E(1))}, \quad (4\text{-}3.1)$$

where $w_E(x) \equiv (2x - M_E - m_E)/(M_E - m_E)$. If we let

$$\gamma \equiv 2/(2 - M_E - m_E), \quad (4\text{-}3.2)$$

$$\sigma_E \equiv (M_E - m_E)/(2 - M_E - m_E) = 1/w_E(1), \quad (4\text{-}3.3)$$

and

$$\rho_{n+1} \equiv 2w_E(1)T_n(w_E(1))/T_{n+1}(w_E(1)), \quad (4\text{-}3.4)$$

then it follows as before that the $P_{n,E}(x)$ satisfy the recurrence relation (4-2.11) with γ and ρ_{n+1} replacing the $\bar{\gamma}$ and $\bar{\rho}_{n+1}$, respectively. Thus from Theorem 3-2.1, the iterates for the polynomial acceleration procedure based on $P_{n,E}(x)$ may be given by

$$u^{(n+1)} = \rho_{n+1}\{\gamma(Gu^{(n)} + k) + (1 - \gamma)u^{(n)}\} + (1 - \rho_{n+1})u^{(n-1)}. \quad (4\text{-}3.5)$$

We remark that the error vector $\varepsilon^{(n)} \equiv u^{(n)} - \bar{u}$ associated with (4-3.5) now satisfies

$$\varepsilon^{(n)} = P_{n,E}(G)\varepsilon^{(0)}. \quad (4\text{-}3.6)$$

Relations analogous to (4-2.15) and (4-2.16) also are valid for ρ_{n+1}; i.e.,

$$\begin{aligned} &\rho_1 = 1, \qquad \rho_2 = (1 - \tfrac{1}{2}\sigma_E^2)^{-1}, \\ &\rho_{n+1} = (1 - \tfrac{1}{4}\sigma_E^2\rho_n)^{-1}, \qquad n \geq 2, \end{aligned} \quad (4\text{-}3.7)$$

and

$$\lim_{n\to\infty} \rho_n \equiv \rho_\infty = \frac{2}{1 + \sqrt{1 - \sigma_E^2}}. \quad (4\text{-}3.8)$$

In the above discussion, we have tacitly assumed that $M_E \neq m_E$. If $M_E = m_E$, then $\gamma = 1/(1 - M_E)$, $\rho_1 = \rho_2 = \cdots = 1$ and $\sigma_E = 0$. For this case, the Chebyshev acceleration procedure (4-3.5) reduces to the (nonoptimum) extrapolation procedure (2-2.13). We shall not consider this case separately in the balance of this section. The correct formulas can be obtained by a limiting process.

It was shown in the previous section that the $P_n(x)$, defined by (4-2.9), is the *unique* polynomial satisfying the inequality (4-2.7). Thus $\bar{\mathbf{S}}(P_{n,E}(G))$ is minimized only when $P_{n,E}(x) \equiv P_n(x)$; i.e., when $m_E = m(G)$ and $M_E = M(G)$. We note that $P_{n,E}(x)$ is that polynomial satisfying $P_{n,E}(1) = 1$ and which has the minimum "maximum" magnitude over the interval $m_E \leq x \leq M_E$. Because of this, equations for $\bar{\mathbf{S}}(P_{n,E}(G))$ and $\bar{R}_\infty(P_{n,E}(G))$ analogous to (4-2.18) and (4-2.22) need not be valid when $P_{n,E}(x)$ is used. In the remainder of this section, we shall study the behavior of $\bar{\mathbf{S}}(P_{n,E}(G))$ as a function of m_E and M_E in more detail.

In order to simplify the discussion while at the same time retaining adequate generality, we will make one of the following two sets of assumptions:†

Case I

$$m_E \leq m(G), \tag{4-3.9}$$

$$M(G) < 1, \tag{4-3.10}$$

$$m_E < M_E < 1. \tag{4-3.11}$$

Case II

$$m_E = -M_E, \tag{4-3.12}$$

$$|m(G)| \leq M(G) < 1, \tag{4-3.13}$$

$$0 < M_E < 1. \tag{4-3.14}$$

In later chapters, we shall strengthen the assumptions of each case by assuming additionally that $M_E < M(G)$.

We now graphically illustrate the behavior of $P_{n,E}(x)$ for two cases, each involving the assumptions of Case I. Figure 4-3.1 shows the behavior of $P_{10,E}(x)$ when $M_E > M(G)$. Here r is defined in (4-3.21). For this case, it is easy to see that

$$\max_{m(G) \leq x \leq M(G)} |P_{10,E}(x)| \leq P_{10,E}(M_E). \tag{4-3.15}$$

The equality holds for the value of $M(G)$ shown in the figure. However, if the value of $M(G)$ were equal to the M^* shown, then strict inequality would hold. Figure 4-3.2 shows the behavior of $P_{10,E}(x)$ when $M_E < M(G)$. Here it is clear that

$$\max_{m(G) \leq x \leq M(G)} |P_{10,E}(x)| = P_{10,E}(M(G)). \tag{4-3.16}$$

† The practicalities of these assumptions are discussed in Section 5.3.

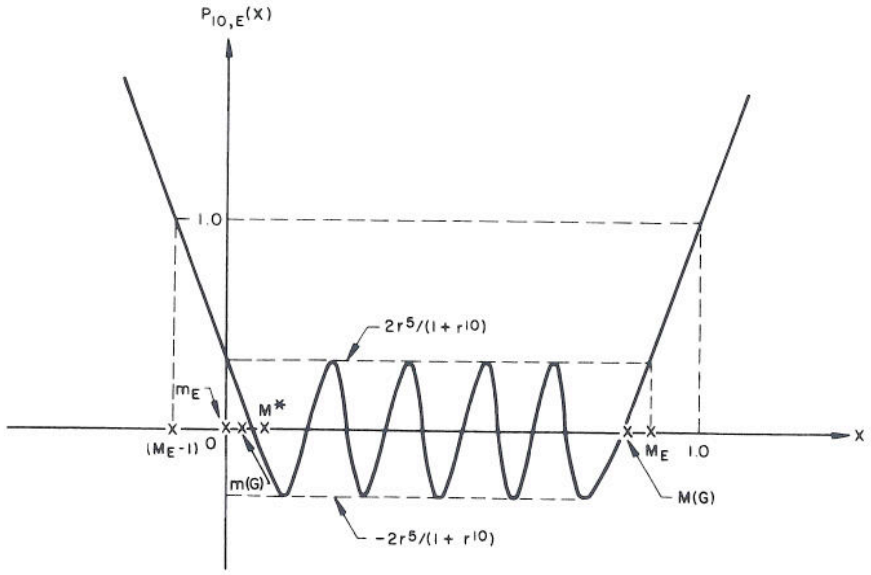

Fig. 4-3.1. Behavior of $P_{10,E}(x)$ for Case I when $M_E > M(G)$. $M_E = 0.94737$, $m_E = 0.00$, $\sigma_E = 0.90$, $r = 0.39$.

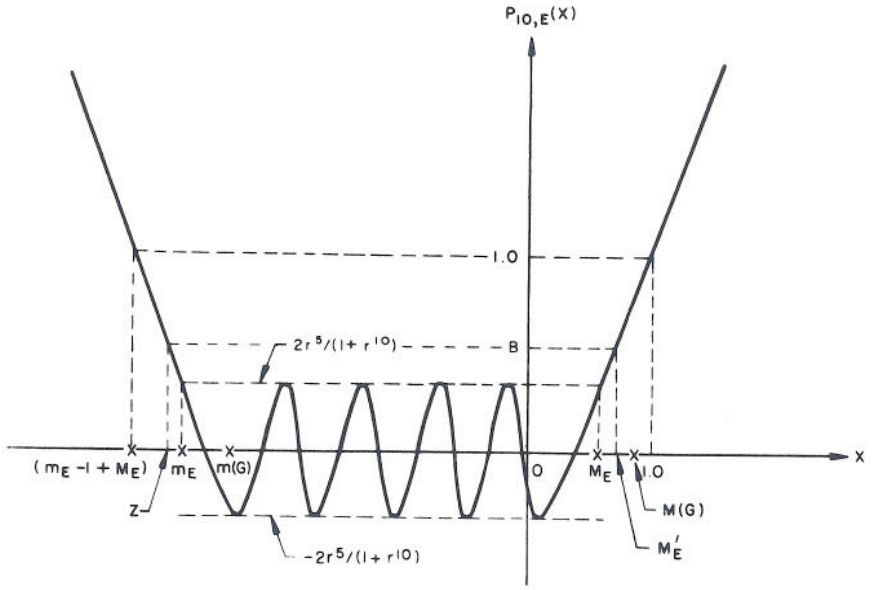

Fig. 4-3.2. Behavior of $P_{10,E}(x)$ for Case I when $M_E < M(G)$. $M_E = 0.789$, $m_E = -3.0$, $\sigma_E = 0.90$, $r = 0.39$.

The symbols B, z, and M'_E shown in Fig. 4-3.2 are used later in Chapter 5.

From the above discussion we have, for either Case I or Case II,† that

$$\begin{aligned} \bar{\mathbf{S}}(P_{n,E}(G)) &= P_{n,E}(M(G)) \quad \text{if} \quad M_E \le M(G), \\ &\le P_{n,E}(M_E) \quad \text{if} \quad M_E \ge M(G). \end{aligned} \tag{4-3.17}$$

It then follows from (4-3.1) that

$$\begin{aligned} \bar{\mathbf{S}}(P_{n,E}(G)) &= T_n\left(\frac{\sigma^*}{\sigma_E}\right) \Big/ T_n\left(\frac{1}{\sigma_E}\right) \quad \text{if} \quad M_E \le M(G), \\ &\le \left[T_n\left(\frac{1}{\sigma_E}\right)\right]^{-1} \quad \text{if} \quad M_E \ge M(G), \end{aligned} \tag{4-3.18}$$

where σ_E is given by (4-3.3) and

$$\sigma^* \equiv (2M(G) - M_E - m_E)/(2 - M_E - m_E). \tag{4-3.19}$$

Note that $w_E(M(G)) = \sigma^*/\sigma_E$ and that $\sigma^*/\sigma_E = 1$ when $M_E = M(G)$. We use σ^*/σ_E instead of $w_E(M(G))$ merely for notation purposes in discussing $\bar{\mathbf{S}}(P_{n,E}(G))$ and $\bar{R}_\infty(P_{n,E}(G))$. Using (4-2.2), we can write the relations (4-3.18) equivalently as

$$\begin{aligned} \bar{\mathbf{S}}(P_{n,E}(G)) &= \frac{2r^{n/2}}{1 + r^n} \Big/ \frac{2\hat{r}^{n/2}}{1 + \hat{r}^n} \quad \text{if} \quad M_E \le M(G), \\ &\le \frac{2r^{n/2}}{1 + r^n} \quad \text{if} \quad M_E \ge M(G), \end{aligned} \tag{4-3.20}$$

where

$$r \equiv \frac{1 - \sqrt{1 - \sigma_E^2}}{1 + \sqrt{1 - \sigma_E^2}}, \qquad \hat{r} \equiv \frac{1 - \sqrt{1 - (\sigma_E/\sigma^*)^2}}{1 + \sqrt{1 - (\sigma_E/\sigma^*)^2}}. \tag{4-3.21}$$

From (4-3.20), it follows that for either Case I or Case II we have

$$\bar{\mathbf{S}}(P_{n,E}(G)) < 1. \tag{4-3.22}$$

From (3-2.17) and (4-3.20), the virtual asymptotic rate of convergence can be given by

$$\begin{aligned} \bar{R}_\infty(P_{n,E}(G)) &= -\log \sqrt{r/\hat{r}} \quad \text{if} \quad M_E \le M(G), \\ &\ge -\log \sqrt{r} \quad \text{if} \quad M_E \ge M(G). \end{aligned} \tag{4-3.23}$$

We now show, under the assumptions of Case II, that the above bound on $\bar{R}_\infty(P_{n,E}(G))$ is an *increasing* function of M_E for $M_E \le M(G)$ and a *decreasing*

† Note that $\bar{\mathbf{S}}(P_{n,I}(G)) = \mathbf{S}(P_{n,I}(G))$ whenever $M_I \le M(G)$.

function of M_E for $M_E \geq M(G)$. Indeed, by (4-3.12), (4-3.3), (4-3.19), and (4-3.21) we have $\sigma_E = M_E$, $\sigma^* = M(G)$,

$$r = \frac{1 - \sqrt{1 - M_E^2}}{1 + \sqrt{1 - M_E^2}}, \tag{4-3.24}$$

and

$$\hat{r} = \frac{1 - \sqrt{1 - (M_E/M(G))^2}}{1 + \sqrt{1 - (M_E/M(G))^2}}. \tag{4-3.25}$$

The fact that $\bar{R}_\infty(P_{n,E}(G))$ is a decreasing function of M_E for $M_E \geq M(G)$ follows from (4-3.23) and (4-3.24). It also follows from (4-3.23) that $\bar{R}_\infty(P_{n,E}(G))$ is an increasing function of M_E for $M_E \leq M(G)$ if we can show that $-\log(r/\hat{r})$ is an increasing function of M_E. But by (4-3.24) and (4-3.25) we have

$$\frac{d}{dM_E}\left[-\log\frac{r}{\hat{r}}\right] = -\frac{2}{M_E}\left\{\frac{1}{\sqrt{1 - M_E^2}} - \frac{1}{\sqrt{1 - (M_E/M(G))^2}}\right\}, \tag{4-3.26}$$

which is positive since $M(G) < 1$. Hence the desired result follows.

We remark that the above result can also be shown to be true for $\bar{R}_n(P_{n,E}(G))$ as well as for $\bar{R}_\infty(P_{n,E}(G))$, and for Case I as well as for Case II.

4.4 SENSITIVITY OF THE RATE OF CONVERGENCE TO THE ESTIMATED EIGENVALUES

In this section, we give quantitative results to illustrate the sensitivity of the asymptotic virtual rate of convergence $\bar{R}_\infty(P_{n,E}(G))$ to the estimates M_E of $M(G)$ and m_E of $m(G)$.

We first consider the behavior of $\bar{R}_\infty(P_{n,E}(G))$ as a function of M_E when $M(G)$ is close to one. For convenience of exposition, we shall assume here that the assumptions of Case II hold with $m(G) = -M(G)$. With these assumptions, we shall show that if $(1 - M(G))$ and $(1 - M_E)$ are both small, then approximately

$$\frac{[\bar{R}_\infty(P_{n,E}(G))]^{-1}}{[\bar{R}_\infty(P_n(G))]^{-1}} \doteq \begin{cases} \sqrt{\theta} + \sqrt{\theta - 1} & \text{if } M_E \leq M(G), \\ \leq 1/\sqrt{\theta} & \text{if } M_E \geq M(G), \end{cases} \tag{4-4.1}$$

where

$$\theta = (1 - M_E)/(1 - M(G)). \tag{4-4.2}$$

For the case $M_E \leq M(G)$, we have by (4-3.23) that

$$\bar{R}_\infty(P_{n,E}(G)) = -\log\sqrt{r} - (-\log\sqrt{\hat{r}}). \tag{4-4.3}$$

But for small $(1 - M_E)$ and small $(1 - M(G))$, by (4-3.24) and (4-3.25) we have, approximately, that

$$r \doteq 1 - 2\sqrt{2}\sqrt{\theta}\sqrt{1 - M(G)}, \qquad \hat{r} \doteq 1 - 2\sqrt{2}\sqrt{\theta - 1}\sqrt{1 - M(G)}. \tag{4-4.4}$$

Thus both r and $\hat{r}$ are close to unity so that $\bar{R}_\infty(P_{n,E}(G))$ of (4-4.3) may be approximated by $\bar{R}_\infty(P_{n,E}(G)) \doteq \frac{1}{2}[(1 - r) - (1 - \hat{r})]$. From this we obtain, using (4-4.4),

$$\bar{R}_\infty(P_{n,E}(G)) \doteq \frac{\sqrt{2}\sqrt{(1 - M(G)}}{\sqrt{\theta} + \sqrt{\theta - 1}}. \tag{4-4.5}$$

Moreover, from (4-2.22), $\bar{R}_\infty(P_n(G)) = -\log\bar{r}^{1/2}$, where $\bar{r} = [1 - (1 - \bar{\sigma}^2)^{1/2}]/[1 + (1 - \bar{\sigma}^2)^{1/2}]$ and where $\bar{\sigma}$ is given by (4-2.16). But $\bar{\sigma} = M(G)$ here since we have assumed that $m(G) = -M(G)$. Combining these facts, we obtain

$$\bar{R}_\infty(P_n(G)) \doteq \tfrac{1}{2}[1 - \bar{r}] \doteq \sqrt{2}\sqrt{1 - M(G)}. \tag{4-4.6}$$

Thus the first part of (4-4.1) follows from (4-4.5) and (4-4.6). A similar argument can be used to show the second part of (4-4.1).

To illustrate these results, we consider the following examples. First, consider the case $M(G) = 0.99$. If $\theta = 1.1$, we have $M_E = 0.989$, which at first sight would seem to be a very close estimate. However, we have $\hat{r} = 0.91400$, $r = 0.74229$, and $\bar{r} = 0.75274$, so that

$$[\bar{R}_\infty(P_{n,E}(G))]^{-1} = 9.61120$$

as compared with

$$[\bar{R}_\infty(P_n(G))]^{-1} = 7.04138.$$

Thus the actual ratio of convergence rates in (4-4.1) is 1.36496. This implies that the expected number of iterations when using $M_E = 0.989$ is 36% more than if $M_E = M(G)$ were used. We note that

$$\sqrt{\theta} + \sqrt{\theta - 1} = 1.3650.$$

Thus the approximation (4-4.1) is quite accurate.

Suppose, on the other hand, that $\theta = 0.9$. In this case, $M_E = 0.991$. Here $r = 0.76388$ and

$$[\bar{R}_\infty(P_{n,E}(G))]^{-1} = (-\tfrac{1}{2}\log r)^{-1} = 7.42544.$$

The left ratio in (4-4.1) is 1.05455. Thus the expected number of iterations, using $M_E = 0.991$, is only about 5.5% more than if $M_E = M(G)$ were used. We note that

$$(\sqrt{\theta})^{-1} = 1.054.$$

Again, the approximation (4-4.1) is quite accurate.

Let us now consider the case $\theta = 2$, in which again $M(G) = 0.99$. In this case, $M_E = 0.98$, $r = 0.66806$, $\hat{r} = 0.75167$, and $[\bar{R}_\infty(P_{n,E}(G))]^{-1} = 16.961$. Thus the left ratio in (4-4.1) is 2.408. This should be compared with the approximation (4-4.1); i.e.,

$$\sqrt{\theta} + \sqrt{\theta - 1} = 2.414.$$

If $1 - M(G)$ is very small, the approximation (4-4.1) is accurate even for quite large θ. Suppose $M(G) = 0.9999$. If $\theta = 100$, then $M_E = 0.99$, $r = 0.75274$, and $\hat{r} = 0.75383$. Moreover, $[\bar{R}_\infty(P_{n,E}(G))]^{-1} = 1382.17$ as compared with $[\bar{R}_\infty(P_n(G))]^{-1} = (-\frac{1}{2}\log \bar{r})^{-1} = 70.706$. (Note that $\bar{r} = 0.97211$.) Thus the left ratio in (4-4.1) is 19.548. This should be compared with the approximation

$$\sqrt{\theta} + \sqrt{\theta - 1} = 19.950.$$

We remark that the behavior of $\bar{R}_\infty(P_{n,E}(G))$ as a function of M_E for Case I conditions is similar to that given above for Case II.

From the above discussion, it is clear that if we underestimate $M(G)$, we increase the expected number of iterations much more than if we overestimate $M(G)$ by an equivalent amount. However, upper bounds for $M(G)$ which are nontrivial in the sense that they are less than unity and yet close to $M(G)$ are very difficult to obtain. Moreover, it is very difficult to improve an overestimated value even if one is available. On the other hand, if M_E is an underestimate for $M(G)$, then improved estimates for $M(G)$ can be obtained by using the adaptive procedures given in Chapters 5 and 6. In addition, as we shall see later, very accurate estimates of the iteration error may be obtained when $M_E < M(G)$. For these reasons, the adaptive procedures described in subsequent chapters are designed so that the estimates M_E converge to $M(G)$ from below.

Turning our attention now to the estimate m_E, we first note that $\bar{R}_\infty(P_{n,E}(G))$ does not depend on m_E when Case II conditions hold. For Case I conditions, we shall show that $\bar{R}_\infty(P_{n,E}(G))$ is relatively insensitive to the choice of m_E as long as $m_E \leq m(G)$.† Specifically, if $(1 - M(G))$ is small, if

† The case in which m_E is greater than $m(G)$ is considered later in Chapter 6.

$M_E = M(G)$, and if $m_E \leq m(G) < 0$, we show that, approximately,

$$\frac{[\bar{R}_\infty(P_{n,E}(G))]^{-1}}{[\bar{R}_\infty(P_n(G))]^{-1}} \doteq \left[\frac{1+\delta-m_E}{1+\delta-m(G)}\right]^{1/2} \leq \left[\frac{1-m_E}{1-m(G)}\right]^{1/2}$$
$$\leq \left[1+\frac{\lambda}{1-m(G)}\right]^{1/2}, \tag{4-4.7}$$

where

$$\delta = 1 - M(G), \qquad \lambda = m(G) - m_E. \tag{4-4.8}$$

Since $M_E = M(G)$, we have from (4-3.23) that $\bar{R}_\infty(P_{n,E}(G)) = -\log r^{1/2}$, where $r = [1-(1-\sigma_E^2)^{1/2}]/[1+(1-\sigma_E^2)^{1/2}]$ and where σ_E is given by (4-3.3). Using (4-4.8) with $M_E = M(G)$, we obtain

$$(1-\sigma_E) = 2\delta(1+\delta-m(G))^{-1}.$$

Since δ is small, it follows that r may be approximated by

$$r \doteq 1 - 2\sqrt{2}\sqrt{1-\sigma_E} = 1 - 2\sqrt{2}\sqrt{2\delta/(1+\delta-m_E)}. \tag{4-4.9}$$

Thus we have approximately that

$$\bar{R}_\infty(P_{n,E}(G)) \doteq \tfrac{1}{2}(1-r) \doteq \sqrt{2}\sqrt{2\delta/(1+\delta-m_E)}. \tag{4-4.10}$$

From (4-2.22) together with (4-2.16) and (4-2.19), it follows similarly that $\bar{R}_\infty(P_n(G))$ may be approximated by

$$\bar{R}_\infty(P_n(G)) \doteq \sqrt{2}\sqrt{2\delta/(1+\delta-m(G))}. \tag{4-4.11}$$

Thus

$$\frac{[\bar{R}_\infty(P_{n,E}(G))]^{-1}}{[\bar{R}_\infty(P_n(G))]^{-1}} \doteq \sqrt{\frac{1+\delta-m_E}{1+\delta-m(G)}} \tag{4-4.12}$$

and (4-4.7) follows.

Thus it is clear from (4-4.7) that Chebyshev acceleration is relatively insensitive to the estimate m_E as long as $m_E \leq m(G)$. For example, if $\lambda(=m(G)-m_E)$ is equal to 0.1, then the expected number of iterations using m_E is only about 4% more than if $m_E = m(G)$ were used. Further, if $m(G) \leq -1.0$, then m_E need satisfy only

$$(m(G)-m_E)/|m(G)| \leq 0.1 \tag{4-4.13}$$

in order that the increase in the number of iterations using m_E be less than 4%.