The Laplacian of a Graph

The Laplacian is another important matrix associated with a graph, and the Laplacian spectrum is the spectrum of this matrix. We will consider the relationship between structural properties of a graph and the Laplacian spectrum, in a similar fashion to the spectral graph theory of previous chapters. We will meet Kirchhoff's expression for the number of spanning trees of a graph as the determinant of the matrix we get by deleting a row and column from the Laplacian. This is one of the oldest results in algebraic graph theory. We will also see how the Laplacian can be used in a number of ways to provide interesting geometric representations of a graph. This is related to work on the Colin de Verdière number of a graph, which is one of the most important recent developments in graph theory.

13.1 The Laplacian Matrix

Let $x$ be an arbitrary orientation of a graph $X$, and let $D$ be the incidence matrix of $X$. Then the Laplacian of $X$ is the matrix $Q(X) = DD^T$. It is a consequence of Lemma 8.3.3 that the Laplacian does not depend on the orientation $x$, and hence is well-defined.

**Lemma 13.1.1.** Let $X$ be a graph with $n$ vertices and $c$ connected components. If $Q$ is the Laplacian of $X$, then $\|Q\| = n - c$.

**Proof.** Let $D$ be the incidence matrix of an arbitrary orientation of $X$. We shall show that $\|Q\| = \|DD^T\| = \|D\|^2$, and the result then follows from Theorem 8.3.1. If $z \in \mathbb{R}^n$ is a vector such that $DD^T z = 0$, then
Note that \( A = I - Q(K_n) \), hence (13.1) can be rewritten as
\[ Q(X) + Q(\overline{X}) = Q(K_n). \]

From the proof of Lemma 13.1.4 it follows that the eigenvalues of \( Q(K_n) \)
are \( n \), with multiplicity \( n - 1 \), and \( 0 \), with multiplicity \( 1 \). Since \( K_n \) is the
complement of \( K_n \), we can see this fact, along with Lemma 13.1.4, to determine the eigenvalues of the complete bipartite graph. We leave
the pleasure of this computation to the reader, noting only the result that the
characteristic polynomial of \( Q(K_{n,n}) \) is
\[ ((1 - x)^{n-1}) \cdot (x^{n-1} - 1). \]

We note another useful consequence of Lemma 13.1.3.

Corollary 13.1.4. If \( X \) is a graph on \( n \) vertices, then \( \lambda_n(X) \leq n \). If \( X \)
has \( s \)-connected components, then the multiplicity of \( n \) as an eigenvalue
of \( Q(X) \) is \( n - s \).

Our last result in this section is a property of the Laplacian that will provide
us with a lot of information about its eigenvalues.

Lemma 13.1.5. Let \( X \) be a graph on \( n \) vertices with Laplacian \( Q \). Then
for any vector \( \mathbf{x} \),
\[ s^T Q s = \sum_{e \in E(X)} (x_e - x_{e'})^2. \]

Proof. This follows from the observations that
\[ s^T Q s = s^T D s - s^T D s = (D s)^T (D s) \]
and that if \( x = (x_e) \), then the entry of \( D^2 s \) corresponding to \( x \) is
\[ (x_e - x_{e'}). \]

13.2 Trees

In this section we consider a classical result of algebraic graph theory, which
states that the number of spanning trees in a graph is determined by the
Laplacian.

First we need some preparatory definitions. Let \( X \) be a graph, and let \( e \) be an edge
of \( X \). The graph \( X \) \( e \)-degenerated is obtained by deleting the edge \( e \) of \( X \). The
graph \( X \) \( \overline{e} \)-reconstructed by identifying the vertices \( u \) and \( v \) and then deleting
\( e \). The graph \( X \) \( \overline{e} \) is obtained by deleting the edge \( e \). The graph \( X \) \( \overline{e} \) is illustrated in Figure 13.1. If a vertex \( x \) is adjacent to both \( e \) and \( \overline{e} \), then there will be multiple edges between \( x \) and the newly identified vertex in \( X \). Furthermore, if \( X \) itself has multiple edges, then any edge between

\[ u \text{ and } v \text{ other than } e \text{ itself becomes loops on the newly identified vertex in } \overline{X}. \]

Depending on the situation, it is sometimes possible to ignore loops, multiple edges, or both.

Figure 13.1. Graph \( \overline{X} \), deletion \( \overline{X} \), and contraction \( X/e \)

If \( M \) is a symmetric matrix with rows and columns indexed by the set
\( E \) and if \( S \subseteq V \), then let \( M(S) \) denote the submatrix of \( M \) obtained by
deleting the rows and columns indexed by elements of \( S \).

Theorem 13.2.1. Let \( X \) be a graph with Laplacian matrix \( Q \). If \( u \) is an arbitrary vertex of \( X \), then \( \det Q(u) \) is equal to the number of spanning trees of \( X \).

Proof. We prove the theorem by induction on the number of edges of \( X \).

Let \( \tau(X) \) denote the number of spanning trees of \( X \). If \( e \) is an edge of \( X \), then every spanning tree either contains \( e \) or does not contain it, so we can count them according to this distinction. There is a one-to-one correspondence between spanning trees of \( X \) that contain \( e \) and spanning trees of \( X/e \), so there are \( \tau(X/e) \) such trees. Any spanning tree of \( X \) that does not contain \( e \) is a spanning tree of \( X \) and so there are \( \tau(X/e) \) of these. Therefore,
\[ \tau(X) = \tau(X/e) + \tau(X/e). \]  \hspace{1cm} (13.2)

In this situation, multiple edges are removed during contraction, but we
may ignore loops, because they cannot occur in a spanning tree.

Now, assume that \( v = u \), and let \( E' \) be the \( v \)-eigen matrix with \( \lambda_v \), and all other entries equal to \( 0 \). Then
\[ Q(u) = Q(X) + \lambda_v + E. \]  \hspace{1cm} (13.3)

from which we deduce that
\[ \det Q(u) = \det Q(X) + \det Q(X/e). \]  \hspace{1cm} (13.4)

Note that \( Q(X)/\lambda_v \) is \( Q(X/e) \).

Assume that in forming \( X/e \) we connected \( v \) with \( e \), so that \( \lambda_{X/e} = \lambda_{X/e} + \lambda_v \). Then \( Q(X/e) \) has rows and columns indexed by \( \lambda_{X/e} \) with the entry being equal to \( Q(u) \), and so we also have that \( Q(u) = Q(X/e) \)
Then we can rewrite (13.3) as
\[ \det Q(\mathbf{x}) = \det Q(\mathbf{X}) \mathbf{r}(\mathbf{a}) = \det Q(\mathbf{X}) \mathbf{r}(\mathbf{a}). \]

By induction, if \( \det Q(\mathbf{X}) \mathbf{r}(\mathbf{a}) = \tau(\mathbf{X}; \mathbf{a}) \) and \( \det Q(\mathbf{X}) \mathbf{r}(\mathbf{a}) = \tau(\mathbf{X}; \mathbf{a}) \), hence
\[ (13.2) \] implies the theorem.

It follows from Theorem 13.3.4 that \( \det Q(\mathbf{a}) \) is independent of the choice of the vertex \( \mathbf{a} \).

**Corollary 13.3.2** The number of spanning trees of \( K_n \) is \( n^{n-2} \).

**Proof.** This follows directly from the fact that \( Q(\mathbf{a}) = \det M(\mathbf{a}) \) for any vertex \( \mathbf{a} \).

If \( M \) is a square matrix, then denote by \( M(i, j) \) the matrix obtained by deleting row \( i \) and column \( j \) from \( M \). The \( ij \)-cofactor of \( M \) is the value
\[ (-1)^{i+j} \det M(i, j). \]

The transpose of \( M \) is called the adjugate of \( M \) and denoted by \( \text{adj} M \). The \( ij \)-entry of \( \text{adj} M \) is the \( ji \)-cofactor of \( M \). The most important property of the adjugate is
\[ M \text{adj}(M) = \text{adj}(M)M = \det(M)I. \]

If \( M \) is invertible, it implies that \( M^{-1} = \text{adj}(M) / \det(M) \). Theorem 13.2.1 implies that if \( Q \) is the Laplacian of a graph, then the diagonal entries of \( \text{adj}(Q) \) are all zero. The full truth is somewhat surprising. All the entries of \( \text{adj}(Q) \) are equal.

**Lemma 13.3.2** Let \( \mathcal{X} \) denote the number of spanning trees in the graph \( X \), and let \( Q \) be the Laplacian of \( X \). Then \( \text{adj}(Q) = \tau(X) \).

**Proof.** Suppose that \( X \) has \( n \) vertices. Assume first that \( X \) is not connected, so that \( \tau(X) = 0 \). Then \( Q \) has rank at most \( n-2 \), so any submatrix of \( Q \) of order \( (n-1) \times (n-1) \) is singular and \( \text{adj}(Q) = 0 \).

Thus we may assume that \( X \) is connected. Then \( \text{adj}(Q) = 0 \) but nonetheless \( \text{adj}(Q) = 0 \). Therefore \( X \) is connected, \( Q \) is regular, and \( \tau(X) \) is equal to the determinant of \( \text{adj}(Q) \). Since \( \text{adj}(Q) \) is symmetric, it follows that it is a constant multiple of \( Q \); hence the result follows from Theorem 13.2.1.

To prove the next result, we need some information about the characteristic polynomial of a matrix. If \( A \) and \( B \) are square \( n \times n \) matrices, then \( \det(A + B) \) may be computed as follows. For each subset \( S \) of \( \{1, \ldots, n\} \), let \( A_S \) be the matrix obtained by replacing the zeros of \( A \) indexed by elements of \( S \) with the corresponding rows of \( B \). Then
\[ \det(A + B) = \sum_S \det A_S. \]

### 13.3 Representations

Define a **representation** of a graph \( X \) in \( \mathbb{R}^m \) to be a map \( \phi : V(X) \to \mathbb{R}^m \) into \( \mathbb{R}^m \). Intuitively, we think of a representation as the positions of the vertices in an Euclidean drawing of a graph. Figure 13.2 shows a representation of the cube in \( \mathbb{R}^3 \).

![Figure 13.2: The cube in \( \mathbb{R}^3 \)](image-url)
We regard the vertices \( \phi(v) \) as row vectors, and thus we may represent \( \phi \) by the \( |V(X)| \times |V(X)| \) matrix \( R \) with the images of the vertices of \( X \) as its rows.

Suppose then that \( \phi \) maps \( V(X) \) into \( \mathbb{R}^n \). We say \( \phi \) is balanced if
\[
\sum_{v \in V(X)} \phi(v) = 0.
\]

Thus \( \phi \) is balanced if and only if \( \Gamma^0 \vec{n} = 0 \). The representation of Figure 13.2 is balanced. A balanced representation has the "centre of gravity" at the origin, and clearly we cannot translate any representation so that it is balanced without losing any information. Henceforth we shall assume that a representation is balanced.

If the columns of the matrix \( R \) are not linearly independent, then the image of \( X \) is contained in a proper subspace of \( \mathbb{R}^n \), and \( \phi \) is just a lower-dimensional representation embedded in \( \mathbb{R}^n \). Any maximal linearly independent subset of the columns of \( R \) would suffice to determine all the properties of the representation. Therefore, we will furthermore assume that the columns of \( R \) are linearly independent.

We can imagine building a physical model of \( X \) by placing the vertices in the positions specified by \( \phi \) and connecting adjacent vertices by identical springs. It is natural to consider a representation to be better if it requires the springs to be less extended. Letting \( || \cdot || \) denote the Euclidean length of a vector \( \mathbf{v} \), we define the energy of a representation \( \phi \) to be the value
\[
E(\phi) = \sum_{(v,w) \in E(X)} ||\phi(v) - \phi(w)||^2,
\]

and hope that natural or good drawings of graphs correspond to representations with low energy. Of course, the representation with least energy is the one where each vertex is mapped to the zero vector. Thus we need to add further constraints, to exclude this.

We can go further by disputing the assumption that the springs are identical. To model this, let \( \omega \) be a function from the edges of \( X \) to the positive real numbers, and define the energy \( E(\phi, \omega) \) of a representation \( \phi \) of \( X \) by
\[
E(\phi, \omega) = \sum_{(v,w) \in E(X)} \omega_{vw} ||\phi(v) - \phi(w)||^2,
\]

where \( \omega_{vw} \) denotes the value of \( \omega \) on the edge \( vw \). Let \( W \) be the diagonal matrix with rows and columns indexed by the edges of \( X \) and with the diagonal entries corresponding to the edges equal to \( \omega_{vw} \).

The next result can be viewed as a considerable generalisation of Lemmas 13.1.5.

**Lemma 13.3.1.** Let \( \phi \) be a representation of the edge-weighted graph \( X \), given by the \( |V(X)| \times |V(X)| \) matrix \( R \). If \( D \) is an associated incidence matrix,
\[
\begin{align*}
\forall \mathbf{v} \in V(X) : & \quad \phi(v) = R\mathbf{v}, \\
\text{Proof.} & \quad \text{The rows of } D^T \text{are indexed by the edges of } X, \text{ and if } (v,w) \in E(X), & \quad \text{then the rows of } D^D = D \text{ are indexed by the faces of } X, \text{ and if } (v,w) \in E(X), & \quad \text{then the rows of } D^T \text{ are indexed by the edges of } X, \text{ and if } (v,w) \in E(X). & \quad \text{Consequently, the diagonal entries of } D^T \text{ and } D^T \text{ have the form } \langle (v,w), (v,w) \rangle. \text{ Consequently, the diagonal entries of } D^T \text{ and } D^T \text{ have the form } \langle (v,w), (v,w) \rangle, \text{ where } \omega_{vw} \text{ is the weight of the edge } (v,w). \end{align*}
\]

Let \( \psi \) be a representation of the edge-weighted graph \( X \), given by the \( |V(X)| \times |V(X)| \) matrix \( R \). If \( D \) is an associated incidence matrix,
13.4 Energy and Eigenvalues

We now show that the energy of certain representations of a graph $X$ are determined by the eigenvalues of the Laplacian of $X$. If $M$ is an invertible $m \times m$ matrix, then the map that sends $x$ to $p(x)M$ is another representation of $X$. This representation is given by the matrix $RM$ and provides as much information about $x$ as does $p$. From this point of view the representation is determined by its column space. Therefore, we may assume that the columns of $R$ are orthogonal to each other, and so show that each column has norm 1. In this situation the matrix $RR^T$ satisfies $RR^T = \lambda_i$, and the representation is called an orthogonal representation.

Theorem 13.4.1. Let $X$ be a graph on $n$ vertices with weighted Laplacian $Q$. Assume that the eigenvalues of $Q$ are $\lambda_1 \leq \cdots \leq \lambda_n$ and that $\lambda_1 > 0$.

The minimum energy of a balanced orthogonal representation of $X$ in $\mathbb{R}^n$ equals $\sum_{i=1}^n \lambda_i$.

Proof. By Lemma 13.3.1 the energy of a representation is $\text{tr}(Q)/n$. From Corollary 9.5.2, the energy of an orthogonal representation in $\mathbb{R}^n$ is bounded below by the sum of the $n$ smallest eigenvalues of $Q$. We can realize this lower bound by taking the columns of $R$ to be vectors $x_1, \ldots, x_n$ such that $Qx_i = \lambda_i x_i$.

Since $\lambda_1 > 0$, we must have $x_1 \neq 0$, and therefore by deleting $x_1$ we obtain a balanced orthogonal representation in $\mathbb{R}^{n-1}$, with the same energy. Conversely, we can reverse this process to obtain an orthogonal representation in $\mathbb{R}^n$ from a balanced orthogonal representation in $\mathbb{R}^{n-1}$ such that these two representations have the same energy. Therefore, the minimum energy of a balanced orthogonal representation of $X$ in $\mathbb{R}^n$ equals the minimum energy of an orthogonal representation in $\mathbb{R}^n$, and this minimum equals $\lambda_1 + \cdots + \lambda_n$. $\square$

This result provides an intriguing algorithmic method for drawing a graph in any number of dimensions. Compute an orthonormal basis of eigenvectors $x_1, \ldots, x_n$ for the Laplacian $Q$ and let the columns of $R$ be $x_1, \ldots, x_n$. Theorem 13.4.1 implies that this yields an orthogonal balanced representation of minimum energy. The representation is not necessarily unique, because it may be the case that $\lambda_{i+1} = \lambda_{i+2}$, in which case there is no reason to choose between $x_{i+1}$ and $x_{i+2}$.

Figure 13.3 shows that such a representation in $\mathbb{R}^2$ can look quite appealing, while Figure 13.4 shows that it may be less appealing.

Both of these graphs are planar graphs on 16 vertices, but in both cases the drawing is not planar. Worse still, in general there is no guarantee that the images of the vertices are even distinct. The representation of the cube in $\mathbb{R}^3$ given above can be obtained by this method.

More generally, any pairwise orthogonal triple of eigenvectors of $Q$ provides an orthogonal representation in $\mathbb{R}^3$, and this representation may have pleasing properties, even if we do not choose the eigenvectors that minimize the energy.

We finish this section with a corollary to Theorem 13.4.1.

Corollary 13.4.2. Let $X$ be a graph on $n$ vertices. Then the minimum value of

$$\sum_{v \in V} (f_v - z_v)^2$$

as $f$ ranges over all $n$-dimensional vectors orthogonal to 1, is $\lambda_1(X)$. The maximum value is $\lambda_n(X)$. $\square$

13.5 Connectivity

Our main result in this section is a consequence of the following bound.

Theorem 13.5.1. Suppose that $N$ is a subgraph of the vertices of the graph $X$. Then $\lambda_i(N) \geq \lambda_i(X \setminus N) + |N|$. 

$\square$
Theorem 13.6.1. Let $X$ be a graph and let $Y$ be obtained from $X$ by adding an edge joining two distinct vertices of $X$. Then

$$\lambda_2(Y) \leq \lambda_2(Y') \leq \lambda_2(Y) + 2.$$ 

Proof. Suppose we get $Y$ by joining vertices $v$ and $w$ of $X$. For any vector $z$, we have

$$\langle z, Q(Y)z \rangle = \sum_{u \in X} (x_v - x_w)^2 = \sum_{v \in X} (x_v - x_w)^2 + \sum_{w \in X} (x_w - x_v)^2.$$ 

If we choose $z$ to be a unit eigenvector of $Q(Y)$, orthogonal to $1$, and with eigenvalue $\lambda_2(Y)$, then by Corollary 13.4.2 we get

$$\lambda_2(Y) \geq \lambda_2(Y) + (x_v - x_w)^2. \tag{13.14}$$

On the other hand, if we take $z$ to be a unit eigenvector of $Q(Y')$, orthogonal to $1$, with eigenvalue $\lambda_2(Y')$, then by Corollary 13.4.2 we get

$$\lambda_2(Y') \leq \lambda_2(Y) + (x_v - x_w)^2. \tag{13.15}$$

It follows from (13.14) that $\lambda_2(Y') \leq \lambda_2(Y)$. We can complete the proof by appealing to (13.15) since $z^2 + \langle z, Q(Y)z \rangle \leq 1$, it is straightforward to see that $(x_v - x_w)^2 \leq 2$, and the result is proved.

A few comments on the above proof. If we add an edge joining the two vertices in $Y_0$ (to get $Y_1$), then $\lambda_2$ increases from $0$ to $2$. Although this example might not be impressive, it does show that the upper bound can be tight. The full proof is indicated in Exercise 8.

Next, we consider what happens to the eigenvalues of $Q(Y)$ when we add an edge to $X$.

Lemma 13.6.1. Let $X$ be a graph and let $Y$ be obtained from $X$ by adding an edge joining two distinct vertices of $X$. Then

$$\lambda_2(Y) \leq \lambda_2(Y) + 2.$$ 

Proof. Suppose we get $Y$ by joining vertices $v$ and $w$ of $X$. For any vector $z$, we have

$$\langle z, Q(Y)z \rangle = \sum_{u \in X} (x_v - x_w)^2.$$ 

If we choose $z$ to be a unit eigenvector of $Q(Y)$, orthogonal to $1$, and with eigenvalue $\lambda_2(Y)$, then by Corollary 13.4.2 we get

$$\lambda_2(Y) \geq \lambda_2(Y) + (x_v - x_w)^2. \tag{13.14}$$

On the other hand, if we take $z$ to be a unit eigenvector of $Q(Y')$, orthogonal to $1$, with eigenvalue $\lambda_2(Y')$, then by Corollary 13.4.2 we get

$$\lambda_2(Y') \leq \lambda_2(Y) + (x_v - x_w)^2. \tag{13.15}$$

It follows from (13.14) that $\lambda_2(Y') \leq \lambda_2(Y)$. We can complete the proof by appealing to (13.15) since $z^2 + \langle z, Q(Y)z \rangle \leq 1$, it is straightforward to see that $(x_v - x_w)^2 \leq 2$, and the result is proved.

A few comments on the above proof. If we add an edge joining the two vertices in $Y_0$ (to get $Y_1$), then $\lambda_2$ increases from $0$ to $2$. Although this example might not be impressive, it does show that the upper bound can be tight. The full proof is indicated in Exercise 8.

Next, we consider what happens to the eigenvalues of $Q(Y)$ when we add an edge to $X$.
Theorem 13.6.2. Let $X$ be a graph with $n$ vertices and let $Y$ be obtained from $X$ by adding an edge joining two distinct vertices of $X$. Then $\lambda_2(X) \leq \lambda_2(Y)$, for all $i$, and $\lambda_n(X) \leq \lambda_n(Y)$, if $i < n$.

Proof. Suppose we add the edge to $X$ to get $Y$. Let $z$ be the vector of length $n$ with a entry $-1$ at vertex $i$ and $1$ at vertex $j$, respectively, and all other entries equal to $0$. Then $Q(Y) = Q(X) + zz^T$, and if we use $Q$ to denote $Q(X)$, we have


By Lemma 8.2.1,

$$\det((H - Q)(I - Q)^T) = 1 - z^T(I - Q)^T(zz^T),$$

and therefore

$$\det((H - Q)(I - Q)^T) = 1 - z^T(I - Q)^Tzz^T.$$

The result now follows from Theorem 8.13.3, applied to the rational function $C(t) = 1 - z^T(I - Q)^Tzz^T$, and the proof of Theorem 9.4.1.

One corollary of this and Theorem 13.4.1 is that if $X$ is a graph and $\lambda_2(X)$ is the largest eigenvalue of the Laplacian matrix of the Petersen graph, then the graph does not have a Hamilton cycle. The eigenvalues of the adjacency matrix of the Petersen graph are $3, 1, -1, -1, -1$, with multiplicities 5, 2, 2, 2, respectively. Therefore, the eigenvalue of the Laplacian matrix of the Petersen graph is $3$, with multiplicity $5$, and $1$ with multiplicity $2$. The eigenvalue of the adjacency matrix of $C_{10}$ is $20$, and for $n = 0, 1, \ldots, 9$, it follows that

$$\lambda_2(C_n) = (3 + 5\sqrt{5})/2 > \lambda_2(P_n) = 2.$$

Consequently, the eigenvalues of the Laplacian matrix of $C_n$ do not intersect the eigenvalues of the Laplacian matrix of the Petersen graph, and therefore the Petersen graph does not have a Hamilton cycle.

We present two further examples in Figure 13.1.1; we can prove that these graphs are not hamiltonian by considering their Laplacian in this back- ward. These two graphs are of some independent interest. They are cubic hypohamiltonian graphs, which are somewhat rare. The first graph, on 18 vertices, is one of the two smallest cubic hypohamiltonian graphs after the Petersen graph. Like the Petersen graph it cannot be 3-edge colored (it is one of the Fischl graphs). The second graph, on 22 vertices, belongs to an infinite family of hypohamiltonian graphs.

It is interesting to note that the technique described in Section 3.2 using the adjacency matrix is not strong enough to prove that these two graphs are not hamiltonian. However, there are cases where the adjacency matrix technique works, but the Laplacian technique does not.

13.7 Conductance and Cutsets

We now come to some of the most important applications of $\lambda_2$. If $X$ is a graph and $S \subseteq V(X)$, let $\delta(S)$ denote the set of edges with one end in $S$ and the other in $V(X) \setminus S$.

Lemma 13.7.1. Let $X$ be a graph on $n$ vertices and let $S$ be a subset of $V(X)$, then

$$\chi_d(S) \leq \frac{\|x_S\|_2}{\sqrt{|S(S \setminus S)|}}.$$  

Proof. Suppose $|S| = n$. Let $z$ be the vector (viewed as a function on $V(X)$ whose values are $-1$ on the vertices in $S$ and $+1$ on the vertices not in $S$. Then $z$ is orthogonal to $1$, so by Corollary 13.4.2

$$\chi_d(S) \leq \frac{\|x_S\|_2}{\sqrt{|S(S \setminus S)|}}.$$

The lemma follows immediately from this.

By way of a simple example, if $S$ is a single vertex with valency $k$, then the Lemma implies that $\chi_d(S) \leq k\sqrt{n}/(n-1)$. This is smaller than Pauly's result, although not by much.

Our next application is much more important. Define the conductance $\Phi(X)$ of a graph $X$ to be the minimum value of

$$\frac{\|x_S\|_2}{\sqrt{|S(S \setminus S)|}}$$

where $S$ ranges over all subsets of $V(X)$ of size at most $|V(X)|/2$. (Many authors refer to this quantity as the spectral sparsity number of a graph. They follow Lovász, which seems odd.) From Lemma 13.7.1 we have at once the following:

Corollary 13.7.2. For any graph $X$ we have $\Phi(X) \geq \lambda_2(X)/2$. 

The real significance of this bound is that \( \lambda_2 \) can be computed to a given accuracy in polynomial time, whereas determining the conductance of a graph is an NP-hard problem. A family of graphs with constant conductance and constant conductance bounded from below by a positive constant is called a family of expanders. These are important in theoretical computer science, fast in practice.

The bottleneck width of a graph on a vertex is the minimum value of \(|S|\) for any subset \( S \) of size \([n/2]\). Again, this is NP-hard to compute, but we do have the following:

**Corollary 13.7.2** The bottleneck width of a graph \( X \) on 2m vertices is at least \(\log_2 X/n\).

We apply this to the cube \(Q_n\). In Exercise 13 it is established that \(\lambda_2(Q_n) = 2\), from which it follows that the bottleneck width of the cube is at least \(2^{-1}\). Since this value is easily realized, we have thus found the exact value.

Let \(\log_2 X\) denote the maximum number of edges in a spanning bipartite subgraph of \(X\). This equals the maximum value of \(|S|\), where \( S \) ranges over all subsets of \( V(X) \) with size at most \( n/2\).

**Lemma 13.7.4** If \( X \) is a graph on \( n \) vertices, then \(\log_2 X \leq n\lambda_2(X)/4\).

**Proof.** By applying Lemma 13.7.1 to the complement of \( X \) we get
\[
|S| \leq |S|_n - |S|_n(X)/n \leq n\lambda_2(X)/4,
\]
which is the desired inequality.

### 13.8 How to Draw a Graph

We will describe a remarkable method, due to Tutte, for determining whether a 3-connected graph is planar.

**Lemma 13.8.1** Let \( S \) be a set of points in \( \mathbb{R}^3 \). Then the vector \( x \) in \( \mathbb{R}^n \) minimizes \( \sum_{i \in S} |x_i - y_i|^2 \) if and only if
\[
x = \frac{1}{|S|} \sum_{i \in S} y_i
\]

**Proof.** Let \( y \) be the centroid of the set \( S \), i.e.,
\[
y = \frac{1}{|S|} \sum_{i \in S} y_i
\]
Then
\[
\sum_{i \in S} (x_i - y_i)^2 = \sum_{i \in S} (x_i - \bar{y} + \bar{y} - y_i)^2
\]
Then \( BR_1 + Q_2 B_2 = 0 \), and so if \( Q_2 \) is invertible, this yields that
\[
R_1 = -Q_2^{-1} B_2 R_1, \quad Y_1 = (Q_1 - B_2^T Q_2 B_2) R_1.
\]

We complete the proof by showing that since \( X \) \( F \) is connected, \( Q_2 \) is invertible. Let \( Y = X \setminus F \). Then there is a nonnegative diagonal matrix \( \Delta_2 \) such that
\[
Q_2 = Q(Y) + \Delta_2.
\]
Since \( X \) is connected, \( \Delta_2 \neq 0 \). We prove that \( Q_2 \) is positive definite. We have
\[
x^T Q_2 x = x^T Q(Y) x + x^T \Delta_2 x.
\]
Because \( x^T Q(Y) x = \sum_{v \in X \setminus F} (r(v) - r)^2 \), we see that \( x^T Q(Y) x \geq 0 \) and that \( x^T Q(Y) x = 0 \) if and only if \( x = 0 \) for some \( r \). But now \( x^T \Delta_2 x = x^T \Delta_1 x \), and this is positive unless \( x = 0 \). Therefore, \( x^T Q_2 x > 0 \) unless \( x = 0 \); in other words, \( Q_2 \) is positive definite, and consequently it is invertible.

Tutte showed that each edge in a 3-connected graph lies in a cycle \( C \) such that no edge not in \( C \) joins two vertices of \( X \setminus C \) is connected. He called these peripheral cycles. For example, any face of a 3-connected planar graph can be shown to be a peripheral cycle.

Suppose that \( C \) is a peripheral cycle of size \( e \) in a 3-connected graph \( X \) and suppose that we are given a mapping \( \eta \) from \( V(C) \) to the vertices of a convex \( e \)-gon in \( \mathbb{R}^2 \) such that adjacent vertices in \( C \) are adjacent in the polygon. It follows from Lemma 13.4.3 that there is a unique isometry \( \rho \) of \( X \) relative to \( F \). This determines a drawing of \( X \) in the plane with all vertices of \( X \setminus C \) inside the image of \( C \). Tutte proved that every planar graph has a drawing with no crossings if and only if \( X \) is planar.

Peripheral cycles can be found in polynomial time, and given this, Lemma 13.4.3 provides an automatic method for drawing 3-connected planar graphs. Unfortunately, from an aesthetic viewpoint, the quality of the output is variable. Sometimes there is a good choice of outside face, maybe a large face as in Figure 13.8, or one that is preserved by an automorphism as in Figure 13.9.

However, particularly if there are a lot of triangular faces, the algorithm tends to produce a large number of faces within faces, many of which are unnecessary.

13.9 The Generalized Laplacian

The rest of this chapter is devoted to a generalization of the Laplacian matrix of a graph. There are many generalized Laplacians associated with each graph, which at first sight seem only remotely related. Nevertheless, certain structural properties of a graph constrain the algebraic properties of its associated matrices. This chapter considers the generalized Laplacian of a graph. The next few sections provide an introduction to the important and recent development.

Let \( X \) be a graph with \( n \) vertices. We call a symmetric \( n \times n \) matrix \( Q \) the generalized Laplacian of \( X \) if \( Q_{ii} = 0 \) when \( i \) and \( j \) are adjacent vertices and \( Q_{jj} = 0 \) when \( i \) and \( j \) are adjacent vertices with no common neighbors. There are two matrices of the form \( Q \) with \( n \) \( n \) entries.

As before, \( A \) is the ordinary Laplacian of a graph, \( \Delta \) is the diagonal matrix of \( A \), and \( D \) is the adjacency matrix of \( X \), then \( -A \) is a generalization of \( A \) and \( Q \) of a generalization of \( Q \).

We will be concerned with the eigenvalues of the \( Q \)-eigenvalues of \( Q \) and if \( Q \) is a generalization of \( A \) then for any \( c \), the matrix \( Q - cI \) is a
13.9 The Generalized Laplacian

Theorem 13.9.1 Let X be a graph with a generalized Laplacian Q. If X is connected, then $\lambda_1(Q) > 0$, and the corresponding eigenvector can be taken to have all its entries positive.

Proof. Choose a constant c such that all diagonal entries of $Q - cl$ are nonpositive. By the Perron-Frobenius theorem (Theorem 8.8.1), the largest eigenvalue of $Q - cl$ is simple and the associated eigenvector may be taken to have only positive entries.

If $x$ is a vector with entries indexed by the vertices of X, then the positive support $\text{supp}_+(x)$ consists of the vertices $u$ such that $x_u > 0$, and the negative support $\text{supp}_-(x)$ of the vector $x$ consists of the vertices $u$ such that $x_u < 0$. A modal domain of $x$ is a component of one of the subgraphs induced by $\text{supp}_+(x)$ or $\text{supp}_-(x)$. A modal domain is positive if it is a component of $\text{supp}_+(x)$, and negative otherwise.

If $y$ is a vector with entries indexed by the vertices of X, then the positive support $\text{supp}_+(y)$ consists of the vertices $v$ such that $y_v > 0$, and the negative support $\text{supp}_-(y)$ of the vector $y$ consists of the vertices $v$ such that $y_v < 0$. A modal domain of $y$ is a component of one of the subgraphs induced by $\text{supp}_+(y)$ or $\text{supp}_-(y)$. A modal domain is positive if it is a component of $\text{supp}_+(y)$, and negative otherwise.

If $y$ is a modal domain of $x$, then $y^2$ is the vector given by

$$y^2 = \begin{cases} 1, & v \in Y \\ 0, & \text{otherwise}. \end{cases}$$

If $Y$ and $Z$ are distinct modal domains with the same sign, then since no edge of $X$ joins vertices in $Y$ to vertices in $Z$,

$$\sum_{v \in Y} Q_{vz} = 0. \quad (13.5)$$

Lemma 13.9.2 Let $x$ be an eigenvector of $Q$ with eigenvalue $\lambda$ and let $Y$ be a positive modal domain of $x$. Then $(Q - \lambda I)x_Y \leq 0$.

Proof. Let $y$ denote the restriction of $x$ to $V(Y)$ and let $z$ be the restriction of $x$ to $V(X) \setminus \text{supp}_+(z)$. Let $Q_Y$ be the submatrix of $Q$ with rows and columns indexed by $V(Y)$, and let $B_Y$ be the submatrix of $Q$ with rows and columns indexed by $V(Y)$ and with columns indexed by $V(X) \setminus \text{supp}_+(x)$. Since $Q_{\lambda} = A$, we have

$$Q_Y v + B_Y z = \lambda y. \quad (13.7)$$

Since $B_Y$ and $z$ are nonnegative, $By$ is nonnegative, and therefore

$$Q_Y v \leq Ay. \quad (13.8)$$

It is not necessary for $x$ to be an eigenvector for the conclusion of this lemma to hold; it is sufficient that $(Q - \lambda I)x \leq 0$. Given our discussion in Section 8.7, we might say that it suffices that $x$ be $\lambda$-eigenharmonic.

Corollary 13.9.3 Let $x$ be an eigenvector of $Q$ with eigenvalue $\lambda$, and let $U$ be the subspace spanned by the vectors $v$, where $v$ ranges over the positive modal domains of $x$. If $v \notin U$, then $Qv = \lambda v$.

13.10 Multiplicities

In this section we show that if $X$ is 2-connected and planar, then $\lambda_2$ has multiplicity at most two, and that if $X$ is 3-connected and planar, then $\lambda_2$ has multiplicity at most three. In the next section we show that there

Proof. If $u = \sum y_i x_i$, then using (13.6), we find that

$$\sum_{v \in V} \sum_{i} y_i Q_{uv} = x^T (Q - \lambda I) x. \quad (13.6)$$

and so the claim follows from the previous lemma.

Theorem 13.10.4 Let $X$ be a connected graph, let $Q$ be a generalized Laplacian of $X$, and let $x$ be an eigenvector for $Q$ with eigenvalue $\lambda_2(Q)$. If $x$ has minimal support, then $\text{supp}_+(x)$ and $\text{supp}_-(x)$ induce connected subgraphs of $X$.

Proof. Suppose that $x$ is a $\lambda_2$-eigenvector with distinct positive modal domains $Y$ and $Z$. Because $X$ is connected, $\lambda_2$ is simple and the sum of $x_Y$ and $x_Z$ contains a vector, say orthogonal to the $\lambda_1$-eigenspace.

Now, $x$ can be expressed as the linear combination of eigenvectors of $Q$ with eigenvalues at least $\lambda_2$, consequently, $x^T (Q - \lambda_2 I) x > 0$ with equality if and only if $x$ is a linear combination of eigenvectors with eigenvalue $\lambda_2$. By the previous lemma, we have $x^T (Q - \lambda_2 I) x = 0$. Therefore, $x$ is an eigenvector of $Q$ with eigenvalue $\lambda_2$ and support equal to $V(Y) \cup V(Z)$.

Any $\lambda_2$-eigenvalue has both positive and negative modal domains, because it is orthogonal to the $\lambda_1$-eigenspace. Therefore, the preceding argument shows that any eigenvector with distinct modal domains of the same sign does not have minimal support. Therefore, since $x$ has minimal support, it must have previously one positive and one negative modal domain.

Lemma 13.10.5 Let $Q$ be a generalized Laplacian of a graph $X$, and let $x$ be an eigenvector for $Q$. Then any vertex not in $\text{supp}(x)$ either has no neighbors in $\text{supp}(x)$, or has neighbors in both $\text{supp}(x)$ and $\text{supp}_+(x)$.

Proof. Suppose that $u \notin \text{supp}(x)$, so $x_u = 0$. Then

$$0 = (Qx)_u = Q_{uv} x_v + \sum_{v \in V} Q_{uv} x_v = \sum_{v \in V} Q_{uv} x_v. \quad (13.9)$$

Since $Q_{uv} = 0$ when $v$ is adjacent to $u$, either $x_u = 0$ or all vertices adjacent to $u$ or the sum has both positive and negative terms. In the former case, $x_u$ is not adjacent to any vertex in $\text{supp}(x)$; in the latter it is adjacent to vertices in both $\text{supp}(x)$ and $\text{supp}_+(x)$.
Lemma 13.10.1 Let $Q$ be a generalized Laplacian for the graph $X$. If $X$ is 3-connected and planar, then no eigenvector of $Q$ with eigenvalue $\lambda_2(Q)$ vanishes on three vertices in the same face of any embedding of $X$.

Proof. Let $\alpha$ be an eigenvector of $Q$ with eigenvalue $\lambda_2$, and suppose that $u$, $v$, and $w$ are three vertices not in supp($\alpha$) lying in the same face. We may assume that $\alpha$ has minimal support and hence supp($\alpha$) is a subset of a connected subgraph $G$ of $X$. Let $\beta$ be a vertex in supp($\alpha$). Since $X$ is 3-connected, Menger's theorem implies that there are three paths in $X$ joining $u$, $v$, and $w$ to $\beta$ (in any order). It follows that there are three vertex-disjoint paths $P_u$, $P_v$, and $P_w$ joining $u$, $v$, and $w$, respectively, to some triple of vertices in $\text{supp}(\alpha)$. Each of these three vertices is also adjacent to a vertex in $N(\text{supp}(\alpha))$. Since each of these paths is 3-connected, the graph $H$ we may construct all vertices in $N(\text{supp}(\alpha))$ to a single vertex, and the vertices in $\text{supp}(\alpha)$ to itself, is isomorphic to the complete graph on 3 vertices. This is impossible.

Corollary 13.10.2 Let $Q$ be a generalized Laplacian for the graph $X$. If $X$ is 3-connected and planar, then $\lambda_2(Q)$ has multiplicity at most three.

Proof. If $\lambda_2 = \lambda_3$, then there is an eigenvector in the associated eigenspace whose support is disjoint from any three given vertices. Thus we conclude that $\lambda_2$ has multiplicity at most three.

The graph $K_3$ is 3-connected and planar. Its adjacency matrix $A$ has eigenvalues $\lambda_1(\lambda_2(\lambda_3))$, both simple, and with multiplicities $\alpha - 2$. Taking $Q = -A$, we see that we cannot drop the assumption that $X$ is 3-connected in the last result.

Lemma 13.10.3 Let $X$ be a 2-connected planar graph with a generalized Laplacian $Q$, and let $\alpha$ be an eigenvector of $Q$ with eigenvalue $\lambda_2(Q)$ and with minimal support. If $u$, $v$, and $w$ are adjacent vertices of a face $F$ such that $\alpha(u) = \alpha(v) = \alpha(w) = 0$, then $\alpha'$ does not vanish on vertices from both the positive and negative support of $\alpha$.

Proof. Since $X$ is 2-connected, the face $F$ is a cycle. Suppose that $F$ contains vertices $p$ and $q$ such that $\alpha(u) = \alpha(v) = \alpha(w) = 0$. Without loss of generality we can assume that they occur in the order $u$, $p$, $q$, and $w$ clockwise around the face $F$, and that the portion of $F$ from $q$ to $p$ contains only vertices not in supp($\alpha$). Let $\alpha'$ be the first vertex not in supp($\alpha$) encountered moving clockwise around $F$ from $q$, and let $\alpha''$ be the first vertex in supp($\alpha$) encountered moving clockwise around $F$ from $u$. Then $\alpha'$, $\alpha''$, and $p$ are distinct vertices of $F$ and occur in that order around $F$. Let $P$ be a path from $\alpha' \to p \to \alpha'' \to q$ of whose vertices other than $\alpha'$ are in supp($\alpha$), and let $N$ be a path from $u$ to $\alpha'$ of whose vertices other than $\alpha'$ are in supp($\alpha$). The existence of the paths $P$ and $N$ is a consequence of Corollary 13.9.4 and Lemma 13.9.5. Because $F$ is a face, the paths $P$ and $N$ and both lie outside $F$, and since their endpoints are vertices of $F$, they must cross. This is impossible, since $P$ and $N$ are vertex-disjoint, and so we have the necessary contradiction.

We call a graph orientable if it has a planar embedding with a face that contains all the vertices. Neither $K_3$ nor $K_4$ is orientable, and it is easy to see that $K_5$ is one of the two graphs. A minor of a graph $X$ is a graph obtained by contracting edges in a subgraph of $X$.

Corollary 13.10.4 Let $X$ be a graph on $n$ vertices with a generalized Laplacian $Q$. If $X$ is 2-connected and orientable, then $\lambda_2(Q)$ has multiplicity at most three.

Proof. If $\lambda_2$ had multiplicity greater than two, then we could find an eigenvector $\alpha$ with eigenvalue $\lambda_2$ such that $\alpha$ vanished on two adjacent vertices in the same face of $X$. However, since $\alpha$ must be orthogonal to its homography.

The tree $K_3$ is orientable, but if $A$ is its adjacency matrix, then $A$ has two eigenvalues, and we cannot drop the assumption in the corollary that $X$ be 2-connected.

13.11 Embeddings

We have seen that if $X$ is a 3-connected planar graph and $Q$ is a generalized Laplacian for $X$, then $\lambda_2(Q)$ has multiplicity at most three. The main result of this section is that if $\lambda_2(Q)$ has multiplicity exactly three, then the representation $\rho$ provided by the hyperplane isometric of $Q$ provides a planar embedding of $X$ on the unit sphere.

As a first step we need to verify that in the case just described, no vertex is mapped to more than one point by $\rho$. This and more follows from the next result.

Lemma 13.11.1 Let $X$ be a 2-connected planar graph with a generalized Laplacian $Q$ such that $\lambda_2(Q)$ has multiplicity three. Let $p$ be a representative vertex of $Q$ whose radicant forms a basis for the hyperplane isometric of $Q$. If $F$ is a face of $X$, remove all its vertices from $Q$ to form the hyperplane isometric of $Q$. Then the images under $\rho$ of any two vertices of $F$ are linearly independent.
Proof. Assume by way of contradiction that $u$ and $v$ are two vertices in a face of $X$ such that $p(u) = p(v)$ for some real number $c$, and let $u, v$ be a third vertex in the same face. Then we can find a linear combination of the columns of $Q$ that vanishes at the vertices $u$, $v$, and $w$, thus contradicting Lemma 13.10.1.

If $p$ is a representation of $X$ that maps no vertex to zero, then we define the normalized representation $\hat{p}$ by

$$\hat{p} = \frac{1}{||p||}p(x).$$

Suppose that $X$ is a 3-connected planar graph with a normalized Laplacian $Q$ such that $\lambda_2(Q)$ has a real eigenvalue. Let $\mu$ be the representation given by $\hat{p}(x)$. By the previous lemma, the corresponding normalized representation $\mu$ is well-defined and maps every vertex to a point on the unit sphere. If $u$ and $v$ are adjacent in $X$, then $\hat{p}(u) \neq \hat{p}(v)$, so there is a unique geodesic on the sphere joining the images of $u$ and $v$. Thus we have a well-defined embedding of the graph $X$ on the unit sphere, and our task is to show that this embedding is planar, i.e., distinct edges can meet only at a vertex.

If $C \subseteq X$, then the convex core generated by $C$ is the set of all non-negative linear combinations of the elements of $C$. A subset of the unit sphere in spherical core is open if and only if it contains all points on any geodesic joining $u$ to $v$. The intersection of the unit sphere with a convex cone in spherical core is convex. Suppose that $F$ is a face in some planar drawing of $X$, and consider the convex core $C$ generated by the images under $\hat{p}$ of the vertices of $F$. This means the unit sphere in a convex spherical polygon, and by Lemma 13.13.1, each edge of $F$ determines an edge of this polygon.

This does not yet imply that our embedding of $X$ on the sphere has no crossings. We next result removes some of the difficulty.

Lemma 13.13.2 Let $X$ be a 2-connected planar graph. Suppose it has a planar embedding where the neighbors of the vertices are, in cycle order, $v_1, v_2, \ldots, v_n$. Let $Q$ be a generalized Laplacian for $X$ such that $\lambda_2(Q)$ has a real eigenvalue. Then the planes spanned by the rows $(x_1, x_2, \ldots, x_n)$ are arranged in the same cycle order around the unit sphere in the plane spanned by $\hat{p}$ as the vertices $v_i$ are arranged in $X$.

Proof. Let $\delta_i$ be an eigenvector with eigenvalue $\lambda_2$ with minimal support such that $\hat{p}(v_i) = \delta_i(v_i) = 1$ (here we are viewing $\hat{p}$ as a function on $V(X)$). By Lemma 13.10.1, we see that neither $\hat{p}(v_j)$ nor $\hat{p}(v_k)$ can be zero, and replacing $\hat{p}$ by $\hat{p} - C\delta_i$, if needed, we may suppose that $\hat{p}(v_1) = 0$. Given this, we prove that $\hat{p}(v_2) = 0$.

Suppose there are some values $k, l$ such that $2 \leq k < l \leq n$ and $\delta_i(v_k) > 0, \delta_i(v_l) > 0$, and $\delta_i(v_j) \leq 0$. Since $\lambda_2(Q)$ is connected, the vertices $v_k$ and $v_l$ are joined in $X$ by a path with all vertices in $\text{supp} (\delta_i)$. Taking $a, b$ along this path, we find a cycle in $X$ that separates $v_k$ from $v_l$. Since $X$ is 2-connected, there are two vertices adjacent to $a$ and $b$ joining $v_k$ and $v_l$ respectively to vertices in $\text{supp} (\delta_i)$. The end vertices of these paths other than $v_k$ and $v_l$ are adjacent to vertices in $\text{supp} (\delta_i)$ and thus have two vertices in $\text{supp} (\delta_i)$ that are adjacent to vertices in $\text{supp} (\delta_i)$. This contradicts the fact that $\text{supp} (\delta_i)$ is connected.

It follows that there is exactly one index $k$ such that $\hat{p}(v_k) = 0$ and $\hat{p}(v_j) > 0$, if $j \neq k$. This follows from Lemma 13.10.5 that $\hat{p}$ has a unique zero in $\text{supp} (\delta_i)$, and therefore $\hat{p}(v_k)$ must be zero.

From this we see that if we choose $v$ such that $\hat{p}(v) = 0$ and $\hat{p}(v_j) > 0$, then there exists a complete graph on $\text{supp} (\delta_i)$, for example the complete graph on two points.

The lemma follows from this.

We now prove that the embedding provided by $\hat{p}$ has no crossings. The argument is topological.

Suppose that $X$ is drawn on a sphere $S_n$ without crossings. Let $S_i$ be a unit sphere, with $X$ embedded on it using $\hat{p}$, as described above. The normalized representation $\hat{p}$ provides an injective map from the vertices of $X$ in $S_i$ to the vertices of $X$ in $S_i$. By Remark 2, we may extend the previous argument to a continuous map $\varphi$ from $S_i$ to $S_i$ which injectively maps each face $F$ in $S_i$ to a spherical convex region on $S_i$. From Lemma 13.11.2, it even follows that $\varphi$ is injective on the union of the faces of $X$ that contain a given vertex. Hence $\varphi$ is a continuous, locally injective map from $S_i$ to $S_i$.

If $X$ is a maximal graph that such a map must be injective, we outline a proof. First, since $X$ is 2-connected and locally injective, there is an integer $k$ such that $\lambda_k(Q) = 0$ for each point $v_i$ in $S_i$. Let $F$ be any graph embedded on $S_i$ with $v_i$ vertices, $v$ edges, and $f$ faces. Then $\varphi^{-1}(F)$ is a plane graph on $S_i$ with $x$ vertices, $y$ edges, and $f$ faces. By Euler's formula,

$$x - y + f - 2 = 0,$$

and therefore $x = y + f - 2$.

Thus we have shown that $\varphi$ is injective, and therefore it is a homeomorphism. We conclude that $\hat{p}$ embeds $X$ on a sphere without crossings.

Exercises

1. If $D$ is the incidence matrix of an oriented graph, then show that any square submatrix of $D$ has determinant 0, 1, or -1.

2. Show that the determinant of a square submatrix of $\text{L}(X)$ is equal to 0 or 1, for some integer $r$. 
3. If M is a matrix, let M(t, j) denote the submatrix we get by deleting row t and column j. Define a 2-face in a graph to be a spanning forest with exactly two components. Let Q be the Laplacian of X. If u, j, and k are vertices of X and u ≠ j ≠ k, show that det Q(u, j, k) is equal to the number of 2-faces with u as one component and j and k in the other.

4. Determine the characteristic polynomial of Q(N, n + 1).

5. An acyclic orientation is an acyclic directed graph with n vertices u such that u has in-degree 0 and each vertex other than u has in-degree 0 and is joined to u by a directed path. (In other words, it is a tree oriented so that all arcs point away from the root.) Let u be a directed graph with adjacency matrix A and let D be the diagonal matrix with all diagonal entries equal to the in-degree of the vertex v. Show that the number of spanning acyclic orientations in Y rooted at a given vertex v is equal to det((D - A)[v, v]).

6. Show that if X is connected and has m vertices, then
   \[\lambda_2(X) = \min_{S \subseteq V(X)} \frac{\sum_{u \in S} \lambda_u}{|S|}\]
   where the minimum is taken over all nonempty subsets S of X.

7. Show that if T is a tree with at least three vertices, then \(\lambda_2(T) \leq 1\), with equality if and only if T is a star (i.e., is isomorphic to K_{1,n}).

8. Let u and v be distinct nonadjacent vertices in the graph X. If \(u \in N(X)\), show that \(\lambda_u(X) = \lambda_u(Y) + 2\) if and only if v is a cut vertex in the graph G.

9. Let D be an oriented incidence matrix for the graph X. Let \(d_i\) denote the valency of the vertex v_i. Show that the largest eigenvalue of D is bounded above by the maximum value of \(d_i\) for any two adjacent vertices u and v in X. Prove this is also an upper bound on \(\lambda_2(X)\). (And for every non-zero vector \(v\), show that the bound is tight if and only if X is bipartite and semi-symmetric.)

10. Let X be a connected graph with \(n\) vertices. Show that there is a subset \(Y \subseteq V(X)\) such that \(2m(Y) \geq n\).

11. Let \(S\) and \(T\) be two graphs on \(n\) vertices with degrees \(d\) and \(\ell\), respectively. Show that \(\lambda_2(S) \leq \lambda_2(T)\) if \(d \leq \ell\).

12. Define a 2-face in a graph to be a spanning forest with exactly two components. Let Q be the Laplacian of X. If u, j, and k are vertices of X and u ≠ j ≠ k, show that det Q(u, j, k) is equal to the number of 2-faces with u as one component and j and k in the other.

13. Use Exercise 12 to show that \(\lambda_2(Q) = 2\).

14. If X is an unoriented graph with diameter \(d\) and volume \(v\), show that \(v(X) \geq \sqrt{d/2}\).

15. Show that a cycle in a connected planar graph is a peripheral cycle if and only if it is a face in every plane embedding of the graph.

16. Let X be a connected graph and let \(\lambda\) be an eigenvalue of Q(X) with eigenvector \(\lambda\). Call a path \(v_0, \ldots, v_n\) strictly decreasing if the values of \(v_i\) on the vertices of the path form a strictly decreasing sequence. Show that if \(\lambda \in \sigma(X)\) and \(\lambda > 0\), then it is joined by a strictly decreasing path to some vertex \(v\) such that \(\lambda(v) < 0\).

17. Let X be a connected graph. Show that if Q(X) has exactly three distinct eigenvalues, then there is a constant \(k\) such that any pair of distinct nonadjacent vertices in X is exactly \(k\) common neighbors. Show further that there is a constant \(p\) such that any pair of distinct nonadjacent vertices in X is exactly \(p\) common neighbors. Find a graph X with this property that is not regular. (A regular graph would be strongly regular.)

18. Let Q be a generalized Laplacian for a connected graph X. If \(\mu\) is an eigenvalue for Q with eigenvector \(\lambda\) and \(\mu\) is a vertex in X such that \(\mu\) is maximal, prove that
   \[\lambda_2(X) = \sum_{\mu \neq \lambda} \lambda_{\mu} < \lambda_2(X)\]

19. Let Q be a generalized Laplacian for a connected graph X and consider the representation \(\mu\) provided by the \(\lambda\)-eigenvalue space. Show that if \(\lambda \in \sigma(X)\) does not lie in the convex hull of the set
   \[\{\mu(x) \in x \in X\}\]
   then there is a vertex \(v\) such that \(\mu(v) = \max_{x \in X} \mu(x)\), for any neighborhood \(v\) of \(v\). (Do not struggle with this grade; a result from optimization.) Define \(T\) to be the convex hull of \(\{\lambda\}\), then
   \[\lambda_2(X) = \mu(x) < \lambda_2(X)\]

20. Let Q be a generalized Laplacian for a path. Show that all the eigenvalues of Q are simple.

21. Let Q be a generalized Laplacian for a connected graph X and let \(\lambda\) be an eigenvector for Q with eigenvector \(\lambda\). Show that if no vertices of \(\lambda\) are zero, then both \(s_{\lambda}\) and \(s_{\lambda}\) are connected.

22. Let Q be a generalized Laplacian for a connected graph X and let \(\lambda\) be an eigenvector for Q with eigenvector \(\lambda\). Show that all the eigenvalues of Q are simple.
First, let's consider the theorem stated in the notes:

**Theorem:** A graph $G$ is connected if and only if there exists a set of vertices $S$ such that for every pair of vertices $u$ and $v$ in $G$, there exists a path from $u$ to $v$ that contains at least one vertex of $S$. 

**Proof:**

1. **Implication ($\Rightarrow$):**
   - If $G$ is connected, then for any two vertices $u$ and $v$, there exists a path $P$ connecting $u$ and $v$.
   - Let $S$ be the set of vertices along the path $P$.
   - For any other path $Q$ from $u$ to $v$, the intersection of $P$ and $Q$ will contain at least one vertex of $S$.

2. **Implication ($\Leftarrow$):**
   - If there exists a set $S$ with the property described, then for any two vertices $u$ and $v$, there exists a path from $u$ to $v$ containing at least one vertex of $S$.
   - Therefore, $G$ is connected.

**References**