15-859N — Spectral Graph Theory — Fall 2009

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Assignment 2 Due date: November 5, 2009

1 Bounding Separators Sizes using Path Embeddings

[10 points] Show that an separator for the d dimensional mesh with n nodes has size $\Omega(n^{(d-1)/d})$ for d fixed using path embeddings.
Hint: Give a lower bound for separators for the complete graph and use this to bound the mesh.

2 A Bad Example for Spectral Partitioning

[10 points]

Define **threshold spectral partitioning** of a possibly weighted graph $G = (V, E)$ to be the vertex partition one gets by; 1) Finding the eigenvector x for λ_2, 2) Sorting the vertices by their value in x, and 3) Returning the best threshold cut.

Let P_n^ϵ, for n even, be the weighted path graph on n vertices where all the edges have unit weight except the middle one which has weight ϵ.

Consider the Cartesian product $M_{cn}^\epsilon = P_{c\sqrt{n}} \otimes P_{\sqrt{n}}^\epsilon$

1. Show that threshold spectral partitioning on the graph M_{cn}^ϵ will generate a quotient cut of size $\Omega(1/\sqrt{n})$ for c sufficiently large and $\epsilon = 1/\sqrt{n}$ while the best cut is of size $O(1/n)$.
2. How big does c need to be for this to happen?
3. Show how to extend these results to the case when one uses the eigenvector from the normalized Laplacian.

3 Chebyshev Polynomials

[10 points]

In this problem we will develop some important identities for Chebyshev Polynomials.

1. Consider the following matrix:

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

   Give a one sentence explanation why $A(\theta)$ is rigid counter clockwise rotation by θ degrees and A_θ^n is a rotation by $n\theta$ degrees.

2. We can abstract A_θ to a matrix $A = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ where c and s are variables in some polynomial ring such that $c^2 + s^2 = 1$. Show that

$$A^n = \begin{pmatrix} T_n(c) & -sQ_n(c) \\ sQ_n(c) & T_n(c) \end{pmatrix}$$

where T_n and Q_n are polynomials in c satisfying:

$$\begin{aligned} T_0(c) &= 1 \\ T_1(c) &= c \\ T_{n+1}(c) &= cT_n(c) - (1-c^2)Q_n(c) \end{aligned}$$

and

$$\begin{aligned} Q_0(c) &= 0 \\ Q_1(c) &= 1 \\ Q_{n+1}(c) &= cQ_n(c) + T_n(c) \end{aligned}$$

3. Use these identities to show that:

$$\begin{aligned} T_{n+1}(c) &= 2cT_n(c) - T_{n-1}(c) \\ Q_{n+1}(c) &= 2cQ_n(c) - Q_{n-1}(c). \end{aligned}$$

   Thus T and Q are Chebyshev Polynomials of the first and second kind respectively. Explain why all the roots of T_n and Q_n lie in the interval $[-1, +1]$ and in this interval T and Q return values in this interval.

4. Show how to diagonalize A for $|c| \geq 1$.

5. Use this diagonal form to show that

$$T_n(c) = \frac{(c+\sqrt{c^2-1})^n + (c-\sqrt{c^2-1})^n}{2} = \frac{(c+\sqrt{c^2-1})^n + (c+\sqrt{c^2-1})^{-n}}{2}$$

4 Fill for Planar Nested Dissection

[10 points]

In a now infamous paper Miller showed that every triangulated planar graph has a simple cycle C of size at most $\sqrt{8n}$ such that the number of vertices in each side of C is at most $2/3n$.

Show how to use this separator to generate a nested dissection ordering in the sense of Gilbert and Tarjan which has at most $O(n \log n)$ fill.

Hint: The main use of the simple cycle to to restrict which vertices can have a fill edge between them. Thus it may be necessary for you to add edge to the planar graphs.

Hint: What fill edges will there be between a top level separator vertex and any other vertex? In particular, show that the average degree of a node in the top level separator is $O(\sqrt{n})$.

Show how to get an operation count of $O(n^{3/2})$.