15-859N — Spectral Graph Theory — Fall 2009

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Assignment 1 Due date: October 20, 2009

1 Eigenvalues of Cartesian Products

[10 points]

Let $G = (V, E, w)$ and $H = (V', E', w')$ be two non-negatively weight graphs. Let $G \otimes H = (\bar{V}, \bar{E}, \bar{w})$ be their Cartesian product, where:

- The vertices are $\bar{V} = V \times V'$
- The edges are $\bar{E} = \{< (x, x'), (y, y') > \quad | \quad [x = y \wedge (x', y') \in E'] \vee [x' = y' \wedge (x, y) \in E]\}$
- What should the edge weights be?

1. Show that the eigenvalues of $G \otimes H$ are the direct sum of those of G and H. That is if the eigenvalues of G are $\{\lambda_1, \ldots, \lambda_n\}$ and those of H are $\{\mu_1, \ldots, \mu_m\}$ the those of $G \otimes H$ are $\{\lambda_i + \mu_j \quad | \quad 1 \leq i \leq n 1 \leq j \leq m\}$
2. Show that the eigenvectors of $G \otimes H$ are the direct product of those of G and H.

2 The Eigenvalues of C_n^k, Odd and Even Eigenvectors

[10 points]

Use the characterization of eigenvalues for direct products to give an asymptotic estimate for λ_2 for the Cartesian product C_n^k of k cycles of length n. Use your estimate to determine the mixing rate for a random walk on C_n^k. You should get the same estimate as in Lovász's survey paper.

3 Eigenvectors for graphs with symmetries

[10 points]

Let $G = (V, E, w)$ be a graph, τ an automorthism, and (λ, x) an eigenvector of L, Lapacian of G.

- Show the vector $< x_{\tau(1)}, \ldots, x_{\tau(n)} >$ is an eigenvector of L with eigenvalue λ.

4 Odd and Even Eigenvectors

[10 points] Suppose that G is a graph an automorthism τ of of order two, an envolution. We say that a real valued function f on the vertices is **even** if $f(V) = f(\tau(V))$ and **odd** if $f(V) = -f(\tau(V))$.

1. Show that if G has an automorphism that is an envolution then its Lapacian has a full set of eigenvectors each of which is either even or odd.

5 Estimating λ_2 for a complete balanced binary tree

[10 points]

Determine upper and lower bounds on λ_2 for a matrix A where A is the Laplacian of the complete balanced binary tree on n nodes. You should get a result of the form:

$$\lambda_2 = \Theta(f(n))$$

where $f(n)$ is some function of n.

6 Estimating λ_2 for the exponentially weighted line.

Let EL_n be the line graph with vertices $\{V_0, \ldots, V_n\}$ where the edge (V_i, V_{i+1}) has weight 2^i. Give upper and lower bounds on λ_2 for the normalized Laplacian of EL_n.

For those of you who have a little more of an experimental bend here is a problem.

7 Experimental Problem

[10 points]

This problem can be summarized as "get a graph and tell me a little about it". To be more specific, you should find a graph that you are going to use in experimental work throughout the class (although you can switch graphs later if you become unhappy with your original choice). The graph should be from some "real-world" source. That is, it should not be constructed at random, or come from an algebraic construction. The graph should have at least 1,000 nodes, and preferably at least 10,000. The best way to get a graph is probably to write a script to read it or grab it from the web. Interesting sources include databases of papers (arxiv, ncstrl), fragments of the web (Wikipedia, links internal to a university, etc.). We get is a useful tool for downloading a large portion of the web, but it is overkill since you only need the links. This part could take a lot of work.

Here's what you should report:

1. What is your graph, and where did you get it.
2. How are you storing your graph?
3. How many nodes are there in your graph?
4. How many edges does it have?
5. How many nodes does it have of degree 1?
6. How many nodes does it have of degree 2?
7. What tools are you using?