Counting & Finding Spanning Trees

G = directed graph

H ∈ G is divergent ST with root R if

1) R can reach all of V(G) in H.
2) Indegree(V) = 1 for V ≠ R.
3) Indegree(R) = 0

Def: D is in-degree matrix of G if

\[ D(i,j) = \begin{cases} \text{deg}(i) & \text{if } i = j \\ -\mathcal{K} & \text{#edge from } i \text{ to } j \end{cases} \]
Lemma \( G = (V,E) \) is a divergent tree iff

1) \( D(i,i) = \begin{cases} 0 & \text{if } i = r \\ 1 & \text{o.w} \end{cases} \)

2) det of minor of \( D_{r,r} \) (removing \( r \)th col / row) = 1

(\Rightarrow) 1) Clear

2) After permuting row & col

\( D \) is upper \( \Delta \) & \( r = 1 \)

\( \det(D_{r}) = 1 \)

(\Leftarrow) Suppose false.

\( \Rightarrow \) \( G \) must contain a disconnected component \( C \),

Each node of \( C \) has in degree 1 and zero col sums.

Thus \( \det(D(C)) = 0 \).

\( \Rightarrow \) \( \det(D(G)) = 0 \) contra!
Thm: # divergent ST rooted at $r \equiv \det(D_{v,r})$

WLOG assume $r = v$.

Note: Co) sums of $D$ are zero.

Start by replacing first col with \( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \).

\[ \det(D_{v}) = \det(D_{v,i}) \]

Expand col 2 into subgraphs with fix edge into $V_2$.

\[ D = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix} \]

\[ |D| = \begin{vmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{vmatrix} \]
For matrix expand next col.
Repeat!

We get exact one matrix per indeg 1 subgraph.

\[ G, \quad \text{expand} \quad G_1 \quad G_2 \quad G_3 \quad G_4 \]

Each term in \( G_1 \) iff divergent tree

\[
\begin{cases}
0 & \text{O.W.}
\end{cases}
\]
The undirected Case

\[ G = (V, E) \] undirected

Let \( G' = (V, E') \) where each edge \( [i, i] \) is viewed as \((i, i) \) & \((i, i)\)

Claim \( \#ST(G) = \#ST_r(G') \)

\( f: ST(G) \rightarrow ST_r(G') \)

\( X \rightarrow X' \)

1-1 & onto

Note \( D(G') \equiv \text{laplacian of } G \)
Alg \textbf{WalkTree} \quad \textbf{Inp.:} \quad G = (V, E) \quad \text{connected undirected \& \; weighted} \\
1) \text{Pick} \; s \in V \; \text{arbitrary} \\
2) \text{Do random walk from} \; s \\
3) \text{Collect edge} \; (i, j) \in E \; \text{of first visit to} \; i \\
4) \text{Return tree} \\

Thm \; \textbf{The WalkTree in random ST} \\

\underline{Preliminaries} \\
Random walk on strongly connected directed graphs. \\
\textbf{ST} = \text{Convergent rooted trees} \\
G = (V, P) \quad \sum_i P_{ij} = 1 \quad P_{ij} = W_{ij} \\
W(T) = \prod_{e \in T} W(e)
\( T_i(G) = \text{ST}(G) \) rooted at \( i \)

\( \mathcal{T}(G) = \text{All rooted ST}. \)

**Markov Chain Thm** Let \( \pi_i \) = stationary prob of being at \( i \)

\[
\pi_i = \frac{\sum_{T \in \mathcal{T}} \pi_i(T)}{\sum_{T \in \mathcal{T}} \pi(T)}
\]

\( W = X_0, X_1, \ldots \) a (random) walk.

**Def** \( B_t = \text{(backwards tree at time } t) \)

\( I = \{X_0, \ldots, X_t\} \)

\( \ell(i, t) = \arg \max_{0 \leq u \leq t} \{ X_j = V_i \} \)

\[ E(B_t) = \{(X_{\ell(i, t)}, X_{\ell(i, t)+1}) \} \quad i \in I_i = X_t \]

root is \( X_t \)
$W = (1, 2, 1, 3, 2, 4, 3, 5)$

$B_5$

Note: $\{W\} \nsubseteq V$ then converge ST.

After cover time we have a random walk on rooted tree of $G$

$B_t, B_{t+1}, B_{t+2}$

One recurrent class
\[ \tau_T = \text{stationary prob of } T \]

**Note**

\[ \Pi_i = \lim_{N \to \infty} \frac{1}{N} \sum_{0 \leq t \leq N} P_r(X_t = i) \]

\[ = \lim_{N \to \infty} \frac{1}{N} \sum_0^N P_r(\beta_t \text{ rooted at } i) \]

\[ = \sum_{T \in T_i} \tau_T \]

**To show** \( \tau_T \approx W(T) \)

Let \( T^{(i)} = \text{ST rooted at } i \)

Find precursors of \( T^{(i)} \)

\[ T^{(k)} \xrightarrow{\text{one step}} T^{(i)} \]
$k$ child of $i$ and $\text{Parent}(i)$ in $T^{(k)}$ is in subtree of $k$ in $T^{(i)}$

\[ T^{(k)} = T^{(i)} + [i,j] - [k_{ij} + t] \]

\[ \Gamma(T^{(i)}) = \sum_{(i,j) \in E} \nabla(T^{(i)} + [i,j] - [k_{ij}, i]) P_{k_{ij}} \]

\[ \sum W(T^{(i)} + [i,j] - [k_{ij}, i]) P_{k_{ij}} \]

\[ = \sum \frac{W(T^{(i)})}{P_{k_{ij}}} P_{k_{ij}} \]

\[ = W(T^{(i)}) \]
Def (Forward Tree) $F_t$

$$f(i,t) = \arg \min \left\{ X_j = V_k \right\}_{0 \leq j \leq t}$$

Edges $$\left\{ (X_{f(i,t)}, X_{f(i,t)-1}) \mid i \in I - X_0 \right\}$$

**Thm** Let $c$ = cover time for $G$, stationary = $\Pi_1 \ldots \Pi_n$

$F_c$ = forward tree at time $c$

$$P_r(B_c = \bar{r}) = P_r(F_c = \bar{r}) = \frac{w(T)}{\sum_{T' \in T(G)} w(T')}$$ (starting from stationary)

Since started from stationary

$$P_r(X_0 = V_0, \ldots, X_k = V_k) = P_r(X_0 = V_k, \ldots, X_k = V_0)$$
Thus \( \Pr(R_k = T \mid \mathfrak{r}) = \Pr(F_k = T \mid \mathfrak{r}) \)

pass to limit!

Cor 1 M random walk on undirected \( G = (V, E) \) starting at \( i \)

c is the cover time starting from \( i \)

tree \( F_c \) in uniform random ST rooted at \( i \)

\[
\text{let } W(T) = \frac{d_i}{\prod_{j \in V} d_j} \text{ T rooted at } i
\]

Cor 2 Hypo same as Cor 1 but we return undirected \( F_c \) then get random tree

1-1 correspondence with \( ST \)