Spectral
12/9/09
V2

Counting & Finding Spanning Trees

G = directed graph

$H \subseteq G$ is **divergent ST** with root r if

1) r can reach all of $V(G)$ in H.
2) $\text{indegree}(v) = 1$ for $v \neq r$.
3) $\text{indegree}(r) = 0$

Def D is in-degree matrix of G if

$$D(i,j) = \begin{cases} d_{in}(i) & \text{if } i = j \\ -k = -\#\text{edge from } i \text{ to } j & i \neq j \end{cases}$$

<u>Lemma</u> $G=(V,E)$ is a divergent Tree iff

1) $D(i,i)=\begin{cases} 0 & \text{if } i=r \\ 1 & \text{o.w} \end{cases}$

2) det of minor of $D_{r,r}$ (removing rth col & row) $=1$

($\Rightarrow$) 1) clear

2) After permuting row & col

D is upper Δ & $r=1$

$\det(D_{1,1})=1$

($\Leftarrow$) suppose false.

$\therefore$ G must contain a disconnected component C.

Each node of C has indegree 1 and zero col sums.

Thus $\det(D(C))=0$.

$\Rightarrow \det(D(G))=0$ contra!

Thm # divergent ST rooted at $r \equiv \det(D_{r,r})$

pf WLOG assume $r = v_1$

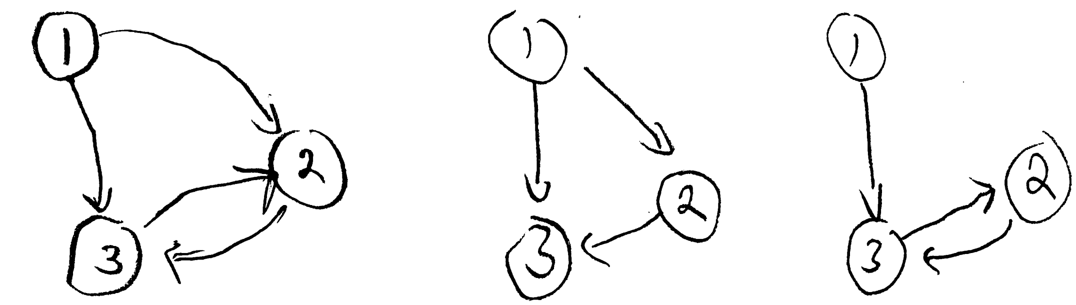
Note: Col sums of D are zero.

Start by replacing first col with $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\det(D) = \det(D_{1,1})$.

Expand col 2 into subgraphs with fix edge into V_2.

eg



G_1 G_2

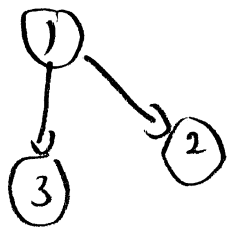
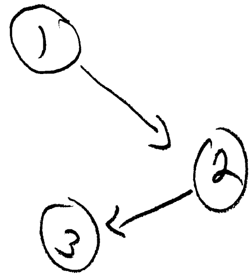
$$D = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

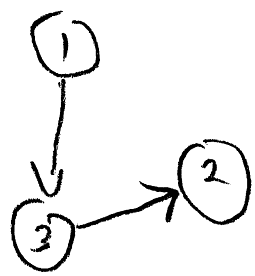
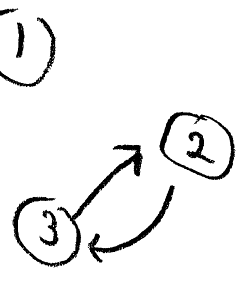
$$|D| = \begin{vmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{vmatrix}$$

For matrix expand next col.

Repeat!

We get exact one matrix per indeg 1 subgraphs.

eg $G_1 \overset{\text{expand}}{\Longrightarrow}$  + 

$G_2 \overset{\text{expand}}{\Longrightarrow}$  

Each term is $\begin{cases} 1 & \text{if } \equiv \text{ divergent tree} \\ 0 & \text{o.w.} \end{cases}$

$\square$

The undirected Case

$G = (V, E)$ undirected

Let $G' = (V, E')$ where each edge $\{i, j\}$ is viewed as (i,j) & (j,i)

<u>Claim</u> $\# ST(G) = \# ST_r(G')$

$f : ST(G) \to ST_r(G')$

1-1 & onto

Note $D(G') \equiv$ Laplacian of G

Random ST by Random Walks

Alg WalkTree InPut: $G=(V,E)$ connected undirected (unit weight)

1) Pick $s \in V$ arbitrary
2) Do random walk from s
3) Collect edge $(i,j) \in E$ if first visit to j
4) Return tree

Thm The WalkTree in random ST

Preliminaries

Random walk on strongly connected directed graphs

ST $\equiv$ convergent rooted trees

$G=(V,P) \quad \sum_j P_{ij}=1 \quad P_{ij}=W_{ij}$

$w(T)=\prod_{e \in T} w(e)$

$\mathcal{T}_i(G) \equiv ST(G)$ rooted at i

$\mathcal{T}(G) \equiv$ All rooted ST.

<u>Markov Chain Thm</u> Let $\pi_i \equiv$ stationary prob of being at i

$$\pi_i = \frac{\sum_{T \in \mathcal{T}_i(G)} w(T)}{\sum_{T \in \mathcal{T}(G)} w(T)}$$

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pf $W = X_0, X_1, \ldots,$ a (random) walk.

<u>Def</u> $B_t \equiv$ (backwards tree at time t)

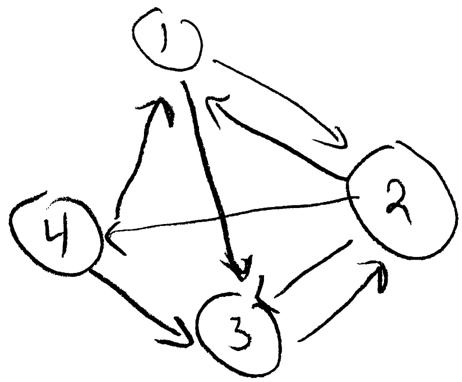
$I = \{X_0, \ldots, X_t\}$

$\ell(i,t) = \underset{0 \le j \le t}{\operatorname{argmax}} \{X_j = V_i\}$

$E(B_t) = \{(X_{\ell(i,t)}, X_{\ell(i,t)+1})\} \quad i \in I - X_t$

root is X_t

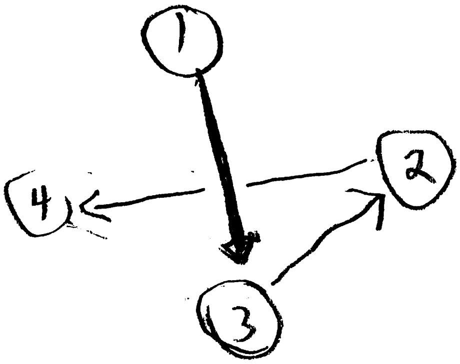
eg



Time 0 1 2 3 4 5

$W = (1, 2, 1, 3, 2, 4, 3 \ldots)$

B_5 root 4



Note: $\{W\} \supseteq V$ then cover ST.

After cover time we have a random walk on rooted tree of G

$B_t, B_{t+1}, B_{t+2}, \ldots$

One recurrent class

$\pi(T) \equiv$ stationary prob of T

Note

$$\pi_i = \lim_{N\to\infty} \frac{1}{N} \sum_{0\le t\le N} P_r(X_t = i)$$

$$= \lim_{N\to\infty} \frac{1}{N} P_r(B_t \text{ rooted at } i)$$

$$= \sum_{T\in T_i} \pi(T)$$

To show $\pi(T) \approx w(T)$

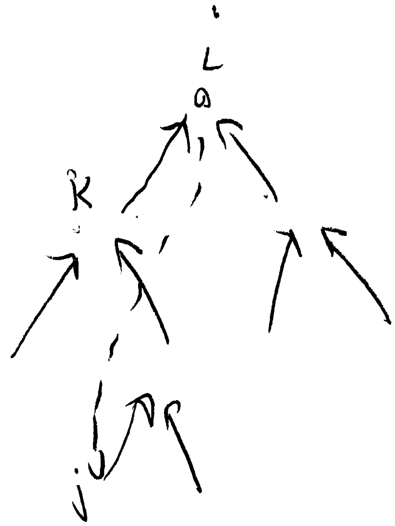
Let $T^{(i)} \equiv$ ST rooted at i

Find precursors of $T^{(i)}$!

$$T^{(k)} \xRightarrow[\text{step}]{\text{one}} T^{(i)}$$

k child of i and Parent(i) in $T^{(k)}$ is in subtree of k in $T^{(i)}$

$T^{(i)}$



$$T^{(k)} = T^{(i)} + [i,j] - [k_j, t]$$

$$\nabla(T^{(i)}) = \sum_{(i,j)\in E} \nabla\left(T^{(i)} + [i,j] - [k_j, i]\right) P_{k_j i}$$

$$\sum W\left(T^{(i)} + [i,j] - [k_i, i]\right) P_{k i}$$

$$= \sum \frac{W(T^{(i)})\, P_{ij}}{\cancel{P_{ki}}} \cancel{P_{ki}} = W(T^{(i)})$$

Def (Forward Tree) F_t

$$f(i,t) = \underset{0 \le j \le t}{\arg\min} \{ X_j = V_i \}$$

Edges $\{ (X_{f(i,t)}, X_{f(i,t)-1}) \mid i \in I - X_0 \}$

Thm Let c = cover time for G, stationary $\equiv \pi_1 \cdots \pi_n$

$F_c \equiv$ forward tree at time c

$$P_r(B_c = T) = P_r(F_c = T) = \frac{w(T)}{\sum_{T' \in T(G)} w(T')} \quad \left(\begin{array}{c} \text{starting from} \\ \text{stationary} \end{array} \right)$$

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pf Since started from stationary

$$P_r(X_0 = V_0, \ldots, X_K = V_K) = P_r(X_0 = V_K, \ldots, X_K = V_0)$$

Thus $\Pr(B_K = T \mid \pi) = \Pr(F_K = T \mid \pi)$

pass to limit!

Cor 1 M random walk on undirected $G = (V, E)$ starting at i

C is the cover time starting from π

Tree F_C is uniform random ST rooted at i

Pf $w(T) = d_i / \prod_{j \in V} d_j$ T rooted at i

Cor 2 Hypo same as Cor 1 but we return

undirected F_C then get random tree.

Pf 1-1 correspondence with ST.