Spectral
12/3/09
r2

Eigens for Tridiagonal Sym Systems

The following method reduce prob to Sym Tri System

$$\left.\begin{array}{l}\text{Jacobi}\\ \text{Givens}\\ \text{Householder}\end{array}\right\}\ n \times n \text{ system}$$

$$\text{Lanczos}\ \}\ m \times m \quad m \ll n$$

$$T = \begin{pmatrix} a_1 & b_1 & & & \\ b_1 & \ddots & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \\ & 0 & \ddots & \ddots & b_{m-1} \\ & & & b_{m-1} & a_m \end{pmatrix}$$

$$P_n(\lambda) = \det(T_n - \lambda I) = \det\begin{pmatrix} a_1-\lambda & b_1 & & \\ b_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & b_{n-1} & a_n-\lambda \end{pmatrix}$$

$P_0 = 1$

$P_1 = a_1 - \lambda$

$P_2 = (a_2-\lambda)P_1(\lambda) - b_1^2$

$P_3 = (a_3-\lambda)P_2(\lambda) - b_2^2\, P_2(\lambda)$

<u>Claim</u> $P_i(\lambda) = (a_i - \lambda)P_{i-1}(\lambda) - b_{i-1}^2\, P_{i-2}(\lambda)$

$P_{-1}(\lambda) = 0 \quad P_0(\lambda) = 1 \quad b_0 = 0$

<u>Thm</u> If $b_1, \dots, b_{n-1} \neq 0$ then roots of P_{i-1} interlace with roots of P_i

ie $\lambda_1 \cdots \lambda_n$ root of P_i

$\lambda_1', \dots, \lambda_{n-1}'$ root of P_{i-1}

$\lambda_1 < \lambda_1' < \lambda_2 \cdots \lambda_{n-1}' < \lambda_n$

<u>Thm</u> If $b_1 \cdots b_{n-1} \neq 0$ then $\forall i \geq 0$

a) $\text{sign}(P_i(-\infty)) = +1$

b) $\text{sign}(P_i(\infty)) = -\text{sign}(P_{i+1}(\infty))$

c) Roots (P_i) interlace Roots(P_{i-1})

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pf

a) case $i=0$ $\quad P_0(\lambda) = 1$ done

b) case $i=0$ $\quad \text{sign}(P_0(\infty)) = 1$

$\text{sign}(P_1(\infty)) = \text{sign}(a_1 - \infty) = -1$ done

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Assume true $i-1$

$$P_i(\lambda) = (a_i - \lambda) P_{i-1}(\lambda) - b_{i-1}^2 P_{i-2}(\lambda)$$

a) $\lambda = -\infty$ $\quad (+\infty)(+) \left(\begin{array}{c} \text{low order} \\ \text{terms} \end{array} \right) = +\infty$ done

b) $\lambda = \infty$ $\quad (-\infty)(\pm\infty)$ $\qquad = \mp\infty$ done

c) Claim root(P_i) distinct roots(P_{i-1})

suppose $\exists \lambda'$ $P_i(\lambda') = P_{i-1}(\lambda') = 0$

$$P_i(\lambda') = (a_i - \lambda') P_{i-1}(\lambda') + b_{i-1}^2 P_{i-2}(\lambda')$$

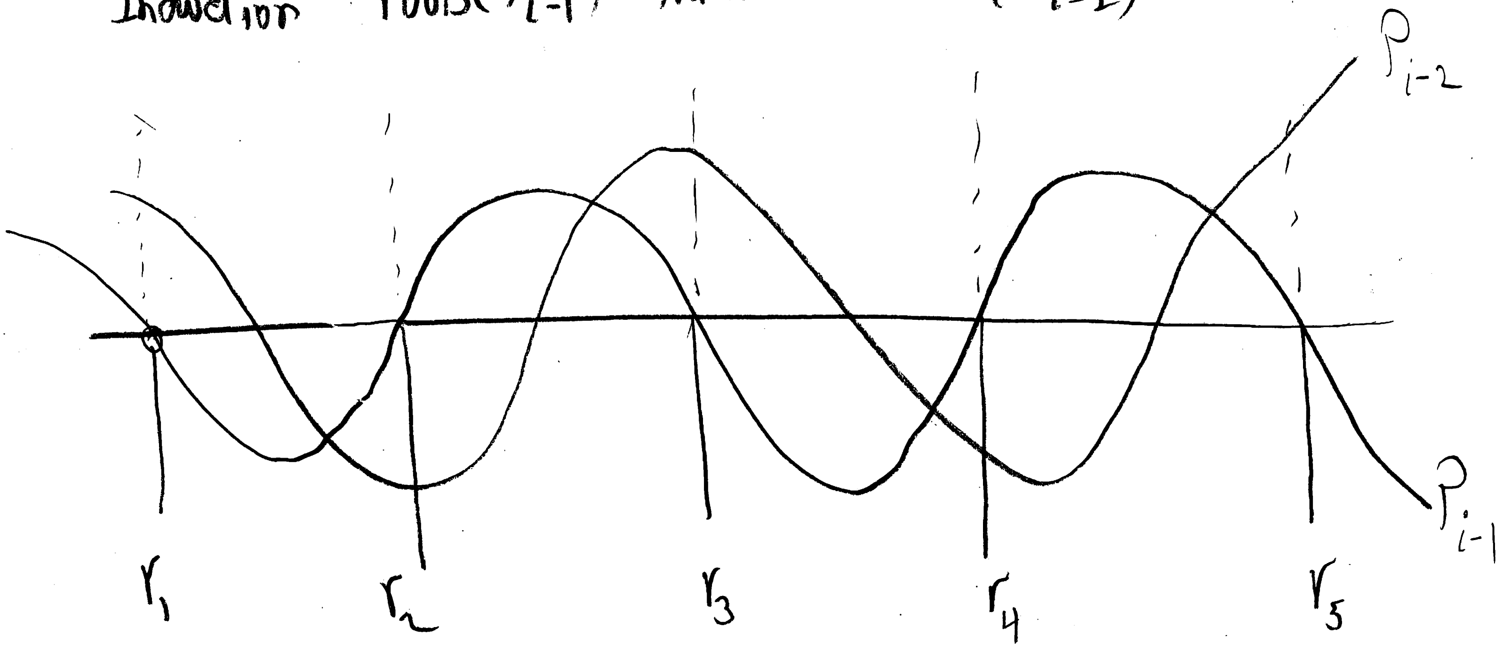
$$\Rightarrow P_{i-2}(\lambda') = 0 \Rightarrow P_0(\lambda') = 0 \text{ contra!}$$

Let $r_1, \dots, r_{i-1}$ be roots of $P_{i-1}(\lambda)$

Thus $P_i(r_k) = -b_{i-1}^2 P_{i-2}(r_k)$

$$\text{sign}(P_i(r_k)) = -\text{Sign}(P_{i-2}(r_k))$$

Induction roots(P_{i-1}) interlace roots(P_{i-2})



Between each pair r_k & r_{k+1} is a root of P_i.

We have found $i-2$ roots of P_i.

By a) & b) $\exists$ root of $P_i < r_1$

$\exists$ $P_i > r_{i-1}$

Thus we have found all the roots of P_i

They interlace!

Using signs to locate roots

Def Given $P_1(x), \dots, P_n(x)$ define

$$N(c) = \left| \{ i \mid \mathrm{Sign}(P_i(c)) \neq \mathrm{Sign}(P_{i+1}(c)) \} \right|$$

Thm # roots of $P_n(x) < c = N(c)$

Cost to compute $N(c)$ is $O(n)$

Divide-and-conquer Alg

$T^{m \times m}$ sym & tridiagonal

$$T = \left(\begin{array}{c|c} T_1 & \beta \\ \hline \beta & T_2 \end{array}\right) = \left(\begin{array}{c|c} \hat{T}_1 & \\ \hline & \hat{T}_2 \end{array}\right) + \left(\begin{array}{c|c} 0 & 0 \\ \quad\;\; \beta & \beta \quad\;\; \\ \hline \quad\;\; \beta & \beta \quad\;\; \\ 0 & 0 \end{array}\right)$$

Suppose we have found eigen vectors for $\hat{T}_1$, $\hat{T}_2$

ie $\hat{T}_1 = Q_1 D_1 Q_1^T \quad \hat{T}_2 = Q_2 D_2 Q_2^T$

Claim

$$T = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \left[\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} + \beta z z^T \right] \begin{bmatrix} Q_1^T & \\ & Q_2^T \end{bmatrix}$$

Where $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ z_1^T = last row of Q_1
z_2^T = first row of Q_2

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to show

$$\beta \left(\begin{array}{c|c} 1 & 1 \\ \hline 1 & 1 \end{array} \right) = \begin{pmatrix} Q_1 & \\ & Q_2 \end{pmatrix} \begin{pmatrix} z\, z^T \end{pmatrix} \begin{pmatrix} Q_1^T & \\ & Q_2^T \end{pmatrix}$$

ie " $\quad$ " $= \begin{pmatrix} Q_1 & \\ & Q_2 \end{pmatrix} \left(\begin{array}{c|c} z_1 z_1^T & z_1 z_2^T \\ \hline z_2 z_1^T & z_2 z_2^T \end{array} \right) \begin{pmatrix} Q_1^T & \\ & Q_2^T \end{pmatrix}$

consider upper-left

$$(\quad _1) = Q_1 z_1 z_1^T Q^T = (Q_1 z_1)(Q_1 z_1)^T$$

$$= \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}^T$$

3-more cases.

We have reduced prob to:

Find eigens of $D + ww^T$ D is diagonal

(assume distinct eigenvalues)

Suppose $w_j = 0$ some j then e_j is an eigenvector.

pf $(D + ww^T) e_j = De_j + ww^T e_j = d_j e_j$

<u>note</u> jth row & col of ww^T is zero.

$\therefore$ $D + ww^T$ is reduceible

Assume $w_j \neq 0 \ \forall j$

<u>Claim</u> $f(\lambda) = 1 + \sum_{j=1}^{m} \frac{w_j^2}{d_j - \lambda}$ then

roots (f) = eigenvalues $(D + ww^T)$.

pf Suppose eigen pair λ, q $q \neq 0$

$$(D + ww^T) q = \lambda q$$

$$(D - \lambda I) q + ww^T q = 0$$

$$q + (D - \lambda I)^{-1} ww^T q = 0$$

mult w^T $$w^T q + w^T (D - \lambda I)^{-1} w (w^T q) = 0$$

$$\underbrace{\left(1 + w^T (D - \lambda I)^{-1} w\right)}_{} (w^T q) = 0$$

$$f(\lambda)(w^T q) = 0$$

Claim $f(\lambda) = 0$ suppose $f(\lambda) \neq 0$ then

$$w^T q = 0 \Rightarrow Dq = \lambda q$$

$$\Rightarrow q = e_j \text{ some } j$$

$$\Rightarrow w_j = 0 \text{ contra!}$$

$$f'(\lambda) = \sum_{j=1}^{m} \frac{d}{d\lambda}\left(w_j^2 (d_j - \lambda)^{-1} \right)$$

$$= \sum_{j=1}^{m} w_j^2 (-1)(d_j - \lambda)^{-2}(-1)$$

$$= \sum w_j^2 (d_j - \lambda)^{-2}$$

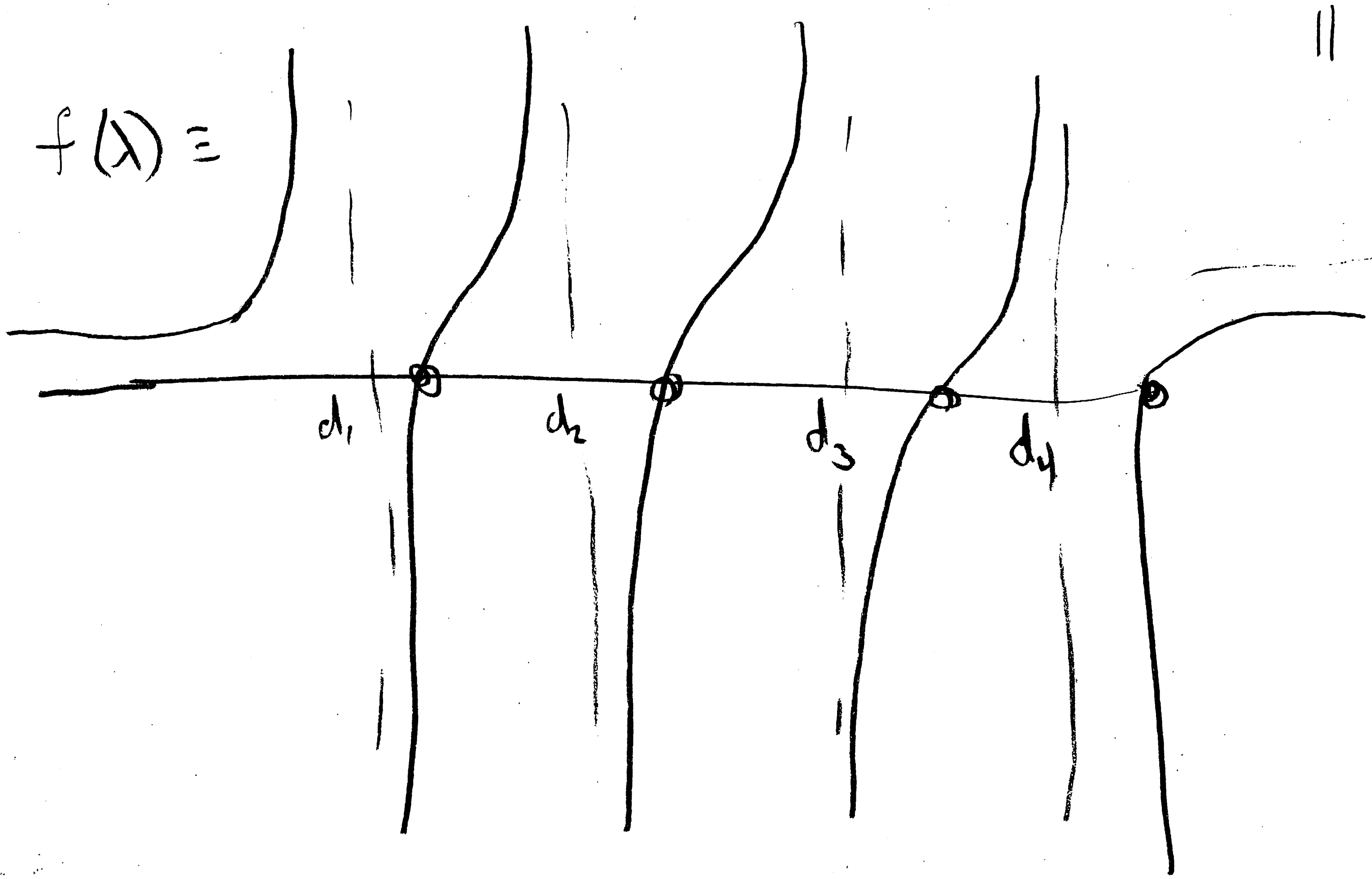
Thus $f'(\lambda) > 0$ for $\lambda \neq$ any d_j

$f(\pm\infty) = 1$

Thus between each d_i, d_{i+1} is a root

d_n, ∞ is a root.

$f(\lambda) \equiv$



Alg 1) Use Newton like scheme to find each root in $O(1)$ iterations & $O(m)$ work/root

$O(m^2)$ total work

$w(m) = m^2 + 2\,w(m/2)$

$w(m) = O(m^2)$

prob: Need (want) eigenvectors.

<u>Thm</u> $A = L(G)$ & $G' = G + e$ $A' = L(G')$

Eigenvalues of A & A' interlace.