Spectral
12/1/09
V2

Spectral Partitioning of Planar Graphs

Let G be a unit weight bounded degree planar graph.

<u>Goal:</u> Show G has an $O(\sqrt{n})$ quotient cut.

$$\phi(G) = O(\sqrt{n}).$$

i.e $\exists S \subseteq V$ s.t. $\dfrac{\text{cut}(S, \bar{S})}{\min\{|S|, |\bar{S}|\}} = O(\sqrt{n})$

By the Cheeger bds

$$\phi(G) \le \sqrt{2 \Delta \lambda_2}$$

Suffice to show $\lambda_2 = O(1/n)$.

Today (ST) $\lambda_2 \le \dfrac{8\Delta}{n}$

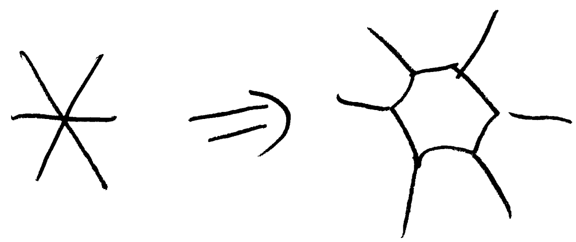
<u>Cor</u> $\phi(G) \le \sqrt{\dfrac{16\Delta^2}{n}} = \dfrac{4\Delta}{\sqrt{n}}$

Bded degree is necessary $\lambda_2(S_n) = \Theta(1)$.

For unbded case we will need vertex sep!

For planar graphs

1) Embed G in the plane
2) replace each vertex with a cycle



3) Find edge separator.
4) remove all orginal vertices common to a separating edge

We need embedding for this construction!

Bdins λ_2

Recall $\lambda_2 = \min_{X^T \mathbb{1} = 0} \dfrac{\sum_{(i,j) \in E} (X_i - X_j)^2}{\sum X_i^2}$

<u>Note</u> $X : V \to \mathbb{R} \wedge X^T \mathbb{1} = 0$ mean $(X) = 0$

<u>Claim</u> $f : V \to \mathbb{R}^d \wedge$ mean$(f) = 0 \wedge d \geq 1$

$$\lambda_2 = \min_f \frac{\sum \| f_i - f_j \|^2}{\sum \| f_i \|^2}$$

<u>pf</u> Suppose $f_i = (X_1^i, \ldots, X_d^i)$

numerator: $\sum (X_1^i - X_1^j)^2 + \sum (X_2^i - X_2^j)^2 + \cdots \sum (X_d^i - X_d^j)^2$

$= X_1^T L X_1 + \cdots + X_d^T L X$

denominator: $X_1^T X_1 + \cdots + X_d^T X_d$

$\text{mean}(f) = 0$ iff $X_1^T \mathbb{1} = 0 \wedge \cdots X_d^T \mathbb{1} = 0$

LHS is just frational aver of d Rayleigh Quotients

Thus $\lambda_2 \leq$ LHS

if f is d-copies of Fedler vector

$\lambda_2 =$ LHS

Goal: Find a "short" edge embedding of G.

We write this as $\dfrac{f^T L f}{f^T f}$

Koebe Embedding

Thm G planar then G has a realization as kissing disk in plane.

ie $\exists$ disks $D_1 \cdots D_n$

1) $int(D_i) \cap int(D_j) = \emptyset$

2) $(i,j) \in E$ iff $D_i \cap D_j \neq \emptyset$

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no pf

Sphere preserving maps.

1) Dilation ie $V \in \mathbb{R}^d \Rightarrow \alpha \cdot V$

2) Ridig motions

3) Reflection about unit sphere

$$X \Rightarrow \frac{X}{X^T X}$$

stereo graphic map.

<u>Claim</u> $\exists$ conformal map of a Koebe embedding st centroid of centers is zero.

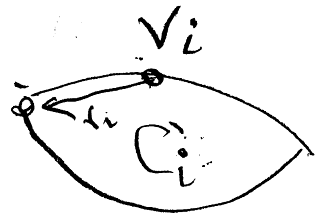
<u>Thm</u> Let f be as in claim then

$$\frac{f^T L f}{f^T f} \leq \frac{8\Delta}{n}.$$

<u>pf</u> Let $C_1 \cdots C_n$ be the caps.

$r_1, \cdots, r_n$ their radii

$v_1 \cdots v_n$ their centers



$$|v_i - v_j| \leq r_i + r_j$$

$$\|v_i - v_j\|^2 \leq (r_i + r_j)^2 \leq 2(r_i^2 + r_j^2) \quad \text{(CS)}$$

$$f^T L f \leq \sum \|v_i - v_j\|^2 \leq 2\Delta \sum r_i^2$$

Caps do not overlap $\Rightarrow$

$$\sum \pi r_i^2 \leq 4\pi \quad \text{ie} \quad \sum r_i^2 \leq 4$$

$$f^T L f \le 8\Delta \qquad \Rightarrow \qquad \frac{f^T L f}{f^T f} \le \frac{8\Delta}{n}$$
$$f^T f = n$$

Thus $\lambda_2 \le \frac{8\Delta}{n}$!

Back to Claim

Def k-ply cap systems $\equiv$ Caps $C_1 \cdots C_n$ on S^d

$\forall x \in S \quad |\{i \mid x \in C_i\}| \le k$.

Assume $k < n/2$ (well behaved)

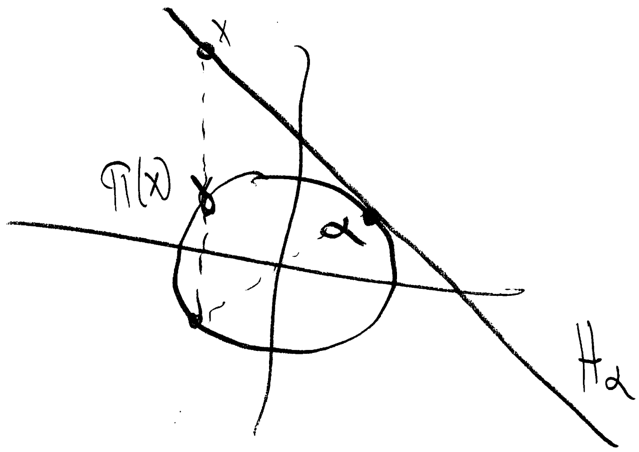
Thm $\forall$ well-behaved cap sys $\{C_1, \cdots, C_n\}$ on S^d

$\exists$ a conformal map f st

centroid of centers $\{f(C_1), \cdots, f(C_n)\} = \vec{0}$

$d=2 \quad \alpha \in S^2$

Π_α: Line-tang-S at $\alpha \Rightarrow S$



$D_\alpha^a \equiv$ dilation of H_α about α by a.

$$f_\alpha^a = \Pi_\alpha^{-1} D_\alpha^a \Pi_\alpha$$

$d=3$

Let σ = centroid$(c(C_1), \dots, c(C_n))$

WLOG $\quad c = (0,0,a) \quad -1 < c \le 0$

Set $\tilde{\alpha} = (0,0,-1)$

$$g_\varepsilon = f_{\tilde{\alpha}}^{1+\varepsilon}$$

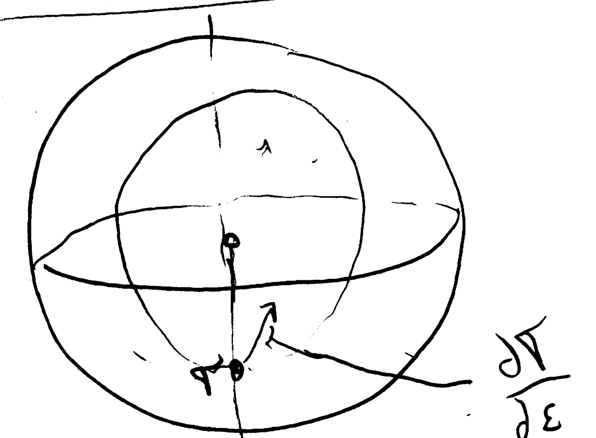
$$\tau_\varepsilon = \text{centroid}\left(c(g_\varepsilon(C_1)), \ldots, c(g_\varepsilon(C_n)) \right)$$

Claim for ε sufficiently small $\|\tau_\varepsilon\| < \|\tau\|$

$$\frac{\partial \tau}{\partial \varepsilon} = \text{centroid}\left(\frac{\partial c(g_\varepsilon(C_1))}{\partial \varepsilon}, \ldots, \frac{\partial c(g_\varepsilon(C_n))}{\partial \varepsilon} \right)$$

note $\dfrac{\partial c(g_\varepsilon(C_i))}{\partial \varepsilon}$ "points up".

except north & south pole which are zeros.



Thus if $\tau \neq 0$ we can always find a small τ.

Stereographic Map

Lets consider a seemingly different map

Reflection about the unit sphere

$$\text{Reflect}(P) = P/P^TP = \frac{P}{|P|^2} \qquad P \in \mathbb{R}^d$$

Note The unit sphere is fixed

inside goes to outside

outside " " inside

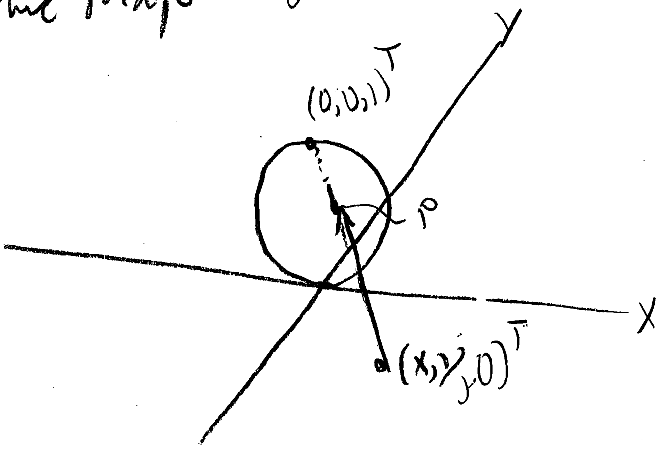
$0 \leftrightarrow \infty$

Claim. Reflect(Reflect(P)) = P

proof
$$\frac{\frac{P}{P^TP}}{\left(\frac{P}{P^TP}\right)^T\left(\frac{P}{P^TP}\right)} = \frac{\frac{P}{P^TP}}{\frac{\cancel{P^TP}}{(P^TP)^{\cancel{2}}}} = P$$

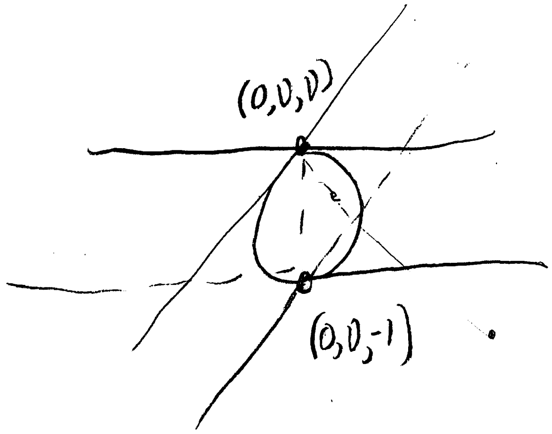
Stereographic Map $d=2$

definition 1



$\text{Ster}(x,y) \to p$

def 2

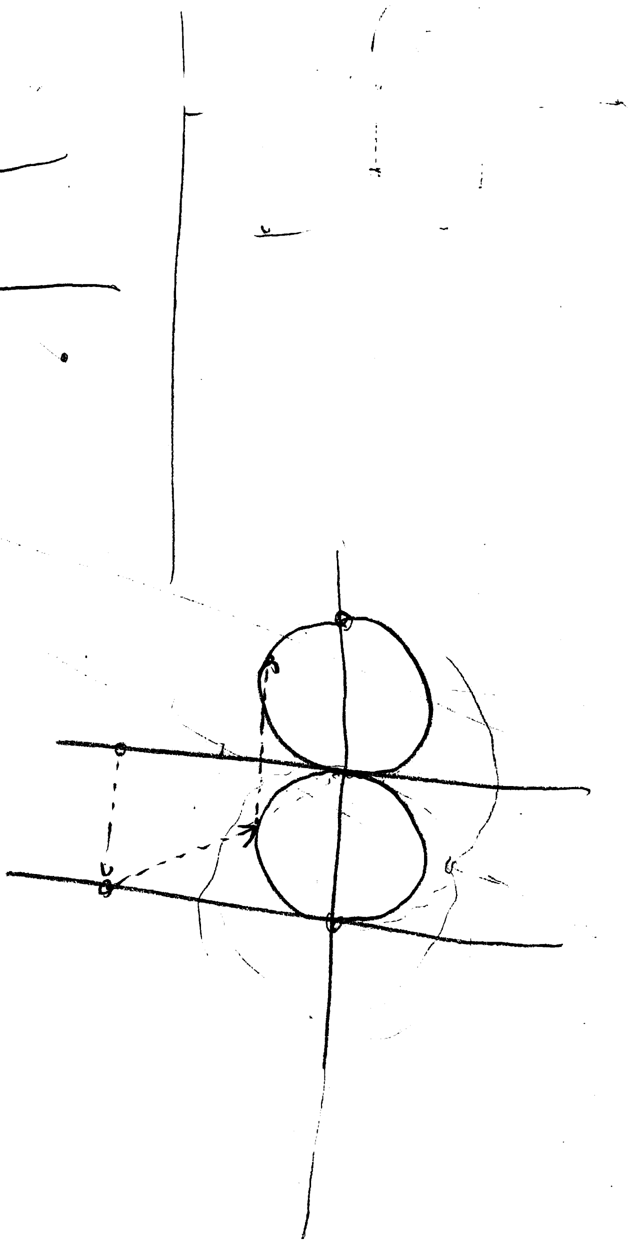


$\text{Stere}(x,y)$

$(x,y) \to (x,y,-1) = p^T$

$p \to p/p^Tp = q$

$q \to q + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$



Claim Spheres go to spheres.

What is a sphere?

Def1 $\{x \mid (x-c)^T(x-c) = r^2\}$

center $c \in R^d$

radius $r \in R^+$

Def2 $\{x \mid \alpha x^T x - 2b^T x + \beta = 0\}$ where $\alpha\beta \leq b^T b$

Claim Def1 plus hyperplane = Def2

1) Def1 $\subseteq$ Def2

$x^T x - 2c^T x + c^T c = r^2$

$x^T x - 2c^T x + c^T c - r^2 = 0$

toshow $c^T c - r^2 \leq c^T c$ ie $r^2 \geq 0$

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2) hyperplane $\subseteq$ Def 2

set $\alpha = 0$

$\{x \mid 2b^T x = \beta\}$

3) Def 2 $\subseteq$ Def 1 + planes

$\alpha = 0$ done

WLOG $\alpha = 1$

$x^T x - 2b^T x + \beta = 0$ s.t. $\beta \leq b^T b$

Just center b, radius $b^T b - \beta$

Claim y satisfies $\alpha y^T y + 2b^T y + \beta = 0$

where $y = \frac{x}{x^T x}$ then $\beta x^T x + 2bx + \alpha = 0$

Pf $\alpha \left(\frac{x}{x^T x}\right)^T \left(\frac{x}{x^T x}\right) + 2b^T \left(\frac{x}{x^T x}\right) + \beta = 0$

$\alpha \left(\frac{1}{x^T x}\right) + 2b^T \left(\frac{x}{x^T x}\right) + \beta = 0$

$\alpha + 2b^T x + \beta(x^T x) = 0$