Spectral
11/24/09

Symmetric Diagonally Dominate Systems

Let A SDD iu $a_{ii} \geq \sum_{i \neq j} |a_{ij}|$

Goal: Reduce SDD to Laplacians!

First idea: Do a change of variables.

$$\text{SDD} \equiv \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ to solve } \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

change of varibles $x' = -x$ & $a' = -a$

$$\text{Solve } \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -x' \\ y \end{pmatrix} = \begin{pmatrix} -a' \\ b \end{pmatrix}$$

$$\text{iu } \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} a' \\ b \end{pmatrix}$$

But this is:
$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} a' \\ b \end{pmatrix}$$

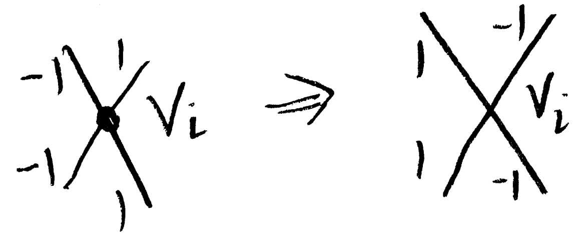
SDD $\equiv$ Graph Laplacians with neg edge weights.

$G = (V, E, w)$ $\quad w(e) \in \mathbb{R}$

Change of variables $x_i' = -x_i$ & $b_i' = -b_i$

Same as negating all edges at V_i

eg



<u>Goal:</u> Make all weights positive by a change of variables.

Let C be a cycle or path in G

<u>Def</u> $\mathrm{sign}(C) = \prod_{e \in C} \mathrm{sign}(w(e))$

<u>Thm</u> All cycles of G have pos sign iff
$\exists$ change of variables with positive edge weights.

($\Leftarrow$) The sign of a cycle is unchange by COV.

($\Rightarrow$) Assume G is connected

pick a vertex $x \in V$

$P(y) = \text{sign}(P_{xy})$

<u>Claim</u> $P(y)$ is well-defind

Let P_{xy} & P'_{xy} be 2 paths from x to y.

$P_{xy} * P'_{yx}$ is a cycle

$1 = \text{sign}(P_{xy} * P'_{yx}) = \text{sign}(P_{xy}) \cdot \text{sign}(P'_{yx})$

$\Rightarrow \text{sign}(P_{xy}) = \text{sign}(P'_{xy})$

If All cycles of G are positive.

we do a COV and solve system.

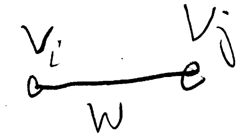
Assume G connected and some neg cycle.

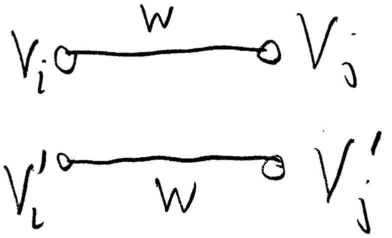
We construct a 2-fold cover.

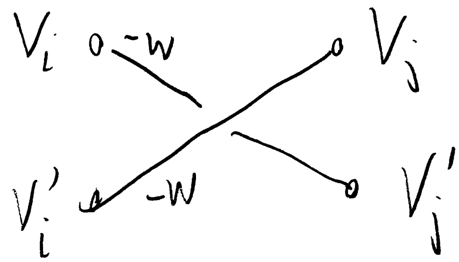
<u>Double Cover</u>

Input: $G = (V, E, w)$

1) $\bar{V} = \{v_1, v_1', v_2 \cdots v_n'\}$

2) for each edge 

a) If $w > 0$ add edges 

b) If $w < 0$ add edges 

Return $\bar{G} = (\bar{V}, \bar{E})$

<u>Claim</u> G connected with neg cycle then

$\bar{G}$ is connected.

<u>pf</u> Suffice to show that v_i connected to v_i'.

iff neg cycle containing v_i

Let C be a neg cycle & P a path from v_i to C

Claim: P, C, P^R is neg.

<u>Cor</u> $L(\bar{G})$ has rank $2n-1$.

$G^+ = (V, E^+, w^+)$

$e \in E^+$ if $e \in E$ & $w(e) > 0$

$G^- = (V, E^-, w^-)$

A^+ adj matrix G^+

A^- " " G^-

$\bar{A} = \begin{pmatrix} A^+ & A^- \\ A^- & A^+ \end{pmatrix}$ adj matrix $\bar{G}$

$L(\bar{G}) = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} - \bar{A} = \bar{L}$

<u>Claim</u> $Lx = b \iff \bar{L}\begin{pmatrix} x \\ -x \end{pmatrix} = \begin{pmatrix} b \\ -b \end{pmatrix}$

<u>Claim</u> $\bar{L}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ -b \end{pmatrix}$ st $\begin{pmatrix} x \\ y \end{pmatrix}^T \mathbb{1} = 0$ then $y = -x$