

Arnoldi Iteration Lanczos Iteration

Spectral
11/19/09
v2

Goal: Compute some of the eigens of A .

Def A similar to A' if $A' = B^{-1}AB$

Eigens of A' are just a change of variables of A .

$\cup$ $Ax = \lambda x$ then $A'z = \lambda z$ for $z = B^{-1}x$

Pl $A'z = B^{-1}AB B^{-1}x = B^{-1}Ax = \lambda B^{-1}x = \lambda z$

Question If A is rational can we find similar rational A' close to diagonal?

Def A is Hessenberg form if

$$H = \begin{array}{|c|} \hline * & & & \\ * & \diagdown & & \\ & & \times & \\ 0 & & & \diagup \\ & & * & * \\ \hline \end{array}$$

Note If A is sym then it is tridiagonal

2-methods we will not do

Jacobi & Householder

Goal: Given A find similar Hessenberg H .

Find Q st $Q^T A Q = H \wedge Q^T Q = I$

ie $AQ = QH$

A is $m \times m$.

We construct Q & H a column at a time.

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$$Q_n = \begin{pmatrix} | & & | \\ q_1 & \dots & q_n \\ | & & | \end{pmatrix} \quad \tilde{A}_n = \begin{matrix} & n \\ m & \begin{array}{|c|} \hline \begin{array}{c} * \quad * \\ * \quad / \\ 0 \quad * \quad * \\ \quad * \end{array} \\ \hline \end{array} \end{matrix} \quad \text{or } \tilde{A}_n \approx (n+1) \times n$$

Suppose we start with b

$$\text{Set } q_1 = b / \|b\|$$

Solve for q_2 & h_{11} & h_{21}

$$V = A q_1 = \begin{pmatrix} | & | \\ q_1 & q_2 \dots \\ | & | \end{pmatrix} \begin{pmatrix} h_{11} \\ h_{21} \end{pmatrix} = h_{11} q_1 + h_{21} q_2$$

$$q_1^T q_2 = 0 \quad q_1^T q_1 = 1$$

$$h_{11} = q_1^T V$$

$$W = V - h_{11} q_1$$

$$h_{21} = \|W\|$$

$$q_2 = W / h_{21}$$



Given Q_n & $\tilde{H}_n$ find Q_{n+1} & $\tilde{H}_{n+1}$

$$A \begin{pmatrix} | & & | \\ q_1 & \dots & q_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ q_1 & \dots & q_{n+1} \\ | & & | \end{pmatrix} \begin{pmatrix} h_{11} & \dots & h_{1,n+1} \\ h_{21} & & \vdots \\ 0 & \dots & h_{n+1,n} \end{pmatrix}$$

Solve $V = A q_n = h_{1n} q_1 + \dots + h_{n+1,n} q_{n+1}$

for $i \leq n$ $q_i^T q_{n+1} = 0$ & $q_{n+1}^T q_{n+1} = 1$

Answer 1) project V onto $\langle q_1, \dots, q_n \rangle$

2) Find orthogonal residual.

Alg Arnoldi Iter

Input A, b

$$q_1 = b / \|b\|$$

for $n=1, 2, \dots$

$$V = A q_n$$

$\vdots$

for $j=1$ to n

$$h_{jn} = q_j^T V$$

$$W = V - \sum_j h_{jn} q_j$$

$$h_{n+1,n} = \|W\| \quad (=0)?$$

$$q_{n+1} = W / h_{n+1,n}$$

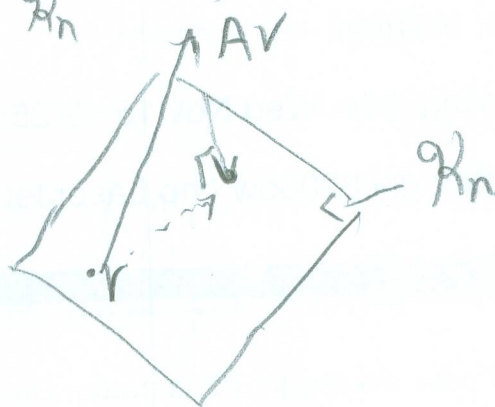
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Claim $\mathcal{R}_n = \langle b, Ab, \dots, A^{n-1}b \rangle = \langle g_1, \dots, g_n \rangle$

$$\|g_i\| = 1 \text{ \& } g_i^T g_j = 0 \text{ for } i \neq j$$

Consider A as acting on $\mathcal{R}_n$

if $v \in \mathcal{R}_n$ $\bar{A}_n(v) = \text{Proj}_{\mathcal{R}_n}(Av)$



a) Work in standard basis

Note $\text{Proj}_{\mathcal{R}_n}(v) = QQ^T v$

$$\bar{A}_n = Q_n Q_n^T A$$

b)

b) Consider $\underbrace{\text{basis}}_{\text{local}} \delta_1, \dots, \delta_n$

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If $w \in \mathcal{H}_n$ in local basis then $v = Q_n w$ is in standard,

$\bar{A}_n$ in local basis is

$$\underbrace{Q_n^T Q_n}_{I} Q_n^T A Q_n (w)$$

$$\bar{A}_n \approx Q_n^T A Q_n$$

What is $Q_n^T A Q_n$?

$$Q_n^T A Q_n = Q_n^T Q_{n+1} \tilde{H}_n = H_n$$

where $H_n = \begin{matrix} & n \\ \begin{matrix} \boxed{H_n} \\ \hline \end{matrix} & \leftarrow H \end{matrix}$

Note $Q_n^T Q_{n+1} = \begin{matrix} & n+1 \\ \begin{matrix} \boxed{\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}} \\ \end{matrix} \end{matrix}$

Thus $\bar{A}_n$ is just A_n .

Def $\lambda(H_n)$ are called Arnoldi eigenvalues or Ritz values.

Note A sym $\Rightarrow A_n$ is tridiagonal.

A_n is a 3-term recurrence

A_n is called Lanczos

Arnoldi & Polynomial Approx

Note $x \in \mathcal{K}_n$ $x = c_0 b + c_1 A b + \dots + c_{n-1} A^{n-1} b$

$$\exists \text{ poly } g(z) = c_0 + c_1 z + \dots + c_{n-1} z^{n-1}$$

$$\text{st } x = g(A)b$$

$$\text{Let } \mathcal{P}^n = \{ \text{monic poly of degree } n \}$$
$$\text{w } c_n = 1$$

Arnoldi / Lanczos Approx Prob

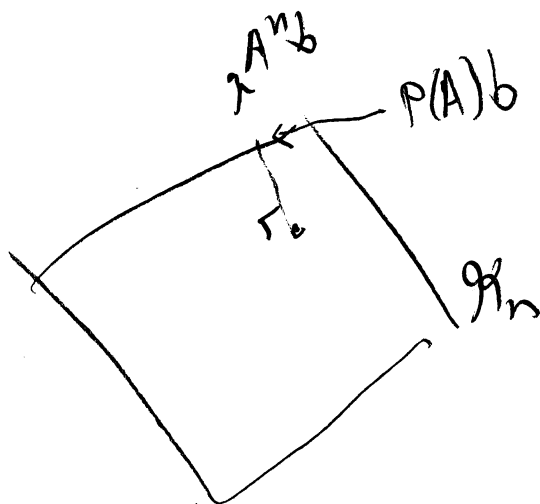
$$\text{Find } p^n \in \mathcal{P}^n \text{ min } \|p^n(A)b\|$$

Thm $\mathcal{K}_n$ not of full rank then p^n is unique
and $p^n = \text{char}(A_n)$

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$$\underline{P^2} \quad P \in P^n \Rightarrow P(A)b = A^n b - Q_n y \text{ for some } y \in \mathbb{R}^n$$

IS $\|P^n(A)b\| \min \Rightarrow P^n(A)b$ is orthogonal complement
of the proj $A^n b$ onto $\mathcal{H}_n$



$$\text{is } Q_n^T P^n(A)b = 0 \quad (1)$$

Claim $P^n(H_n) = 0$

We know $A = QHQ^T$

Thus $Q_n^T P^n(QHQ^T)b = 0$

but $Q_n^T Q P^n(H)Q^T b$

note $Q^T b = \begin{pmatrix} * \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

The first columns of $P^n(H)$ is only dep on first n columns of H

$$\bar{H}_n = \left(\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \begin{matrix} h_1 & \dots & h_n \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{array} \right) = \left(\begin{array}{c|c} H_n & 0 \\ \hline 0 & 0 \end{array} \right) \quad \text{hmm}$$

$$Q_n^T Q P^n(\bar{H}_n) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0 \quad (*)$$

$$Q_n^T Q = \begin{array}{|c|} \hline \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \hline \begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix} \end{array}$$

~~$$(2) \quad P^n(\bar{H}_n) = 0$$~~

~~$$\text{Cayley-Hamilton} \rightarrow P^n = \text{char}(H_n)$$~~

Thus $Q_n^T Q P^n(H_n) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \text{top } n \text{ elements in first column of } P^n(H_n)$

Note $h_{n+1,n}$ only effects the $n+1$ first elements in last column of $P^n(H)$

$$\therefore Q_n^T Q P^n(H) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0 \quad \text{iff} \quad P^n(H_n) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0$$

$$\text{iff first column}(P^n(H_n)) = 0$$

Let $\phi = \text{char}(H_n) \Rightarrow \phi$ is monic

$\phi(H_n) = 0$ by Cayley-Hamilton Thm

Thus $\text{1st-col}(\phi(H_n)) = 0$

So ϕ works but we know it is unique.

QED