

Eigenvalues: Iterative methods

Spectral
11/17/09

Power Iteration $A = A^T$

Alg Init: $\|V^0\| = 1$ (Pick V^0 at random on unit sphere)

for $k=1, 2, \dots$

$$W = AV^{(k-1)}$$

(Sparse matrix-vector product)

$$V^{(k)} = W / \|W\|$$

(normalize)

$$\lambda^{(k)} = V^{(k)T} A V^{(k)}$$

Rayleigh Quotient

Analysis

Init step

Question: Suppose $V = (a_1, \dots, a_n)$ is random point on unit sphere. How big is a_1 ?

$$a_1^2 + \dots + a_n^2 = 1 \Rightarrow E(a_1^2) = 1/n \Rightarrow E(a_1) = \frac{1}{\sqrt{n}}$$

Fact: $\Pr\left[|a_1| \geq \frac{2}{3\sqrt{n}}\right] > .5$

$$\lambda_1 > \lambda_2 \geq \lambda_3 \dots \geq \lambda_n$$

Let g_1, \dots, g_n be orthonormal eigenvectors

$$V^{(0)} = a_1 g_1 + \dots + a_n g_n$$

Consider PI without normalizing

$$\lambda^{(k)} = \frac{V^{(k)T} A V^{(k)}}{V^{(k)T} V^{(k)}} = (*)$$

$$V^{(k)} = a_1 \lambda_1^k g_1 + \dots + a_n \lambda_n^k g_n$$

$$V^{(k)T} V^{(k)} = a_1^2 \lambda_1^{2k} + \dots + a_n^2 \lambda_n^{2k}$$

$$V^{(k)T} A V^{(k)} = a_1^2 \lambda_1^{2k+1} + \dots + a_n^2 \lambda_n^{2k+1}$$

$$* = \frac{\sum a_i^2 \lambda_i^{2k+1}}{\sum a_i^2 \lambda_i^{2k}}$$

$$\lambda_1 - \lambda^{(k)} = \frac{\lambda_1 (\sum a_i^2 \lambda_i^{2k}) - \sum a_i^2 \lambda_i^{2k+1}}{\sum a_i^2 \lambda_i^{2k}}$$

$$= \frac{(\sum_{i>1} a_i^2 \lambda_i^{2k}) (\lambda_1 - \lambda_i)}{a_1 \lambda_1^{2k} + \sum_{i>1} a_i^2 \lambda_i^{2k}}$$

$$\leq \frac{(\sum a_i^2 \lambda_2^{2k})(\lambda_1 - \lambda_2)}{a_1^2 \lambda_1^{2k} + \sum a_i^2 \lambda_2^{2k}}$$

$$\leq \frac{(\lambda_1 - \lambda_2) \lambda_2^{2k}}{a_1^2 \lambda_1^{2k}} \leq O\left(\left(\frac{\lambda_2}{\lambda_1}\right)^{2k}\right)$$

EG Using Power Iter. for Fiedler vector

$$L = L(P_n) \quad \lambda(L) \approx 0, \frac{1}{n^2}, \frac{1}{(n/2)^2}, \frac{1}{(n/3)^2}, \dots, 2$$

We want $\lambda_2 \approx \frac{1}{n^2}$

$L = D - A$ suppose we power $A \approx 2D - L$

$$\begin{aligned} \lambda_{\max} &\approx 2 - \frac{1}{n^2} & \lambda_{\text{next}} &\approx 2 - \frac{4}{n^2} \\ &= \frac{2n^2 - 1}{n^2} & &= \frac{2n^2 - 4}{n^2} \end{aligned}$$

$$\frac{\lambda_{\text{next}}}{\lambda_{\max}} \approx \frac{2n^2 - 4}{2n^2 - 1}$$

Thus we will need $\approx n$ iteration

Inverse Iteration

We have fast solver so lets use it.

Inverse Iteration

Init: Pick random $\|V^0\| = 1 \wedge (V^0)^T \cdot \mathbf{1} = 0$

for $i=1$ to $k = \frac{C}{\epsilon} \log\left(\frac{Cn}{\epsilon}\right)$

Solve $AV^{(i)} = V^{(i-1)}$

Return $V^{(k)}$ & $\lambda^{(k)} = \frac{V^{(k)T} A V^{(k)}}{V^{(k)T} V^{(k)}}$

$$\lambda(A) = 0 = \mu_1 < \mu_2 \leq \dots \leq \mu_n$$

$$r = V^{(0)} = \alpha_2 g_2 + \dots + \alpha_n g_n$$

$$V^{(k)} = \alpha_2 \mu_2^{-k} g_2 + \dots + \alpha_n \mu_n^{-k} g_n$$

$$V^{(k)T} V^{(k)} = \sum \frac{\alpha_i^2}{\mu_i^{2k}}$$

$$\lambda^{(k)} = \frac{V^{(k)T} A V^{(k)}}{V^{(k)T} V^{(k)}} = \frac{\sum_{i \geq 2} \alpha_i^2 / \mu_i^{2k-1}}{\sum_{i \geq 2} \alpha_i^2 / \mu_i^{2k}} = (*)$$

Pick $j = \max_i \{M_i \leq (1 + \epsilon/8) M_2\}$

$$(*) \leq \frac{\sum_{j \geq i \geq 2} \alpha_i^2 / \mu_i^{2k-1}}{\sum_{j \geq i \geq 2} \alpha_i^2 / \mu_i^{2k}} + \frac{\sum_{i > j} \alpha_i^2 M_i^{2k-1}}{\sum_{i \geq 2} \alpha_i^2 / \mu_i^{2k}}$$

$$\leq U_j + \frac{\sum_{i > j} \alpha_i^2 / \mu_i^{2k-1}}{\alpha_2^2 / \mu_2^{2k}}$$

$$\leq U_j + \frac{M_2}{\alpha_2^2} \sum \alpha_i^2 / \left(\mu_i / \mu_2 \right)^{2k-1}$$

Solve
for
 z

$$\frac{1}{z^2} \left(\frac{1}{1 + \frac{\epsilon}{8}} \right)^{\frac{8}{\epsilon} z} \leq \frac{\epsilon}{8}$$

$$\text{LHS} \leq \frac{9n}{4} \left(\frac{1}{1 + \frac{\epsilon}{8}} \right)^{\frac{8}{\epsilon} z} \leq \frac{9n}{4} e^{-z} \stackrel{?}{\leq} \frac{\epsilon}{8}$$

$$\left(\frac{8}{\epsilon} \right) \left(\frac{9n}{4} \right) \leq e^z$$

$$\frac{18n}{\epsilon} \leq e^z \Rightarrow z \geq \ln \left(\frac{18n}{\epsilon} \right)$$

$$\text{Set } 2k-1 = \frac{8}{\epsilon} \ln \left(\frac{18n}{\epsilon} \right) \Rightarrow k \approx \frac{4}{\epsilon} \ln \left(\frac{18n}{\epsilon} \right)$$

$$\leq U_j + \frac{M_2}{\alpha_2^2} \left(\frac{M_2}{M_{j+1}} \right)^{2k-1}$$

$$\leq (1 + \varepsilon/8) M_2 + \frac{M_2}{\alpha_2^2} \left(\frac{1}{1 + \varepsilon/8} \right)^{2k-1} \quad (**)$$

With prob $\geq 1/2$ assume $|\alpha_2| \geq \frac{2}{3\sqrt{n}}$ $\alpha_2^2 \geq \frac{4}{9n}$

$$\text{Suppose: } \frac{1}{\alpha_2^2} \left(\frac{1}{1 + \varepsilon/8} \right)^{2k-1} \leq \frac{\varepsilon}{8}$$

$$\leq (1 + \varepsilon/8) M_2 + \left(\frac{\varepsilon}{8} \right) M_2 = (1 + \varepsilon/4) M_2$$

$$\text{Thus } \lambda^{(k)} \leq (1 + \varepsilon/4) M_2$$

Arnoldi Iteration (general matrix)

Lanczos Iter (sym matrix)

Claim Finding roots of poly \leq_P computing eigenvalues

$$P(z) = z^m + a_{m-1}z^{m-1} + \dots + a_0$$

Consider Companion matrix

$$\begin{pmatrix} 0 & & & -a_0 \\ 1 & & & -a_1 \\ & \ddots & & \vdots \\ 0 & & 0 & -a_{m-1} \end{pmatrix}$$

Note $\det \begin{pmatrix} -z & & & -a_0 \\ 1 & & & -a_1 \\ & \ddots & & \vdots \\ & & -z & -a_{m-1} \end{pmatrix} = P(z)$

Iterative methods on Krylov Spaces

Eg Powering picks $Av^{(k)}$ from

$$\mathcal{K}_n = (v^{(0)}, \dots, Av^{(n-1)})$$

Question: Can we pick $x \in \mathcal{K}_n$ to $\max \frac{x^T Ax}{x^T x}$

Question: Can we think of $A: \mathcal{K}_n \rightarrow \mathcal{K}_n$?

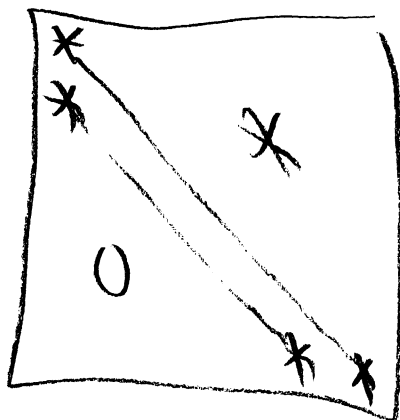
Def $\bar{A}_n(x) = \text{Proj}_{\mathcal{K}_n}(Ax)$

$Q_n = \begin{pmatrix} | & | \\ q_1 & \dots & q_n \\ | & | \end{pmatrix}$ orthonormal basis of \mathcal{K}_n

$$\text{Proj}_{\mathcal{K}_n} = Q_n Q_n^T$$

Eigenvalues are not rational!

Def H is in Hessenberg form if



Goal: A trans to Hessenberg H with same eigen

$$\bar{A}_n = Q_n Q_n^T A$$

∴ \bar{A}_n is linear trans of \mathcal{K}_n with eigenvalues & vectors.

$y \in \mathcal{K}_n$ written in basis $\left(\begin{matrix} | & & | \\ \delta_1 & & \delta_n \\ | & & | \end{matrix} \right)$

$$A_n y = Q_n^T A Q_n y$$

Goal: Find Q_n st $Q_n^T A Q_n$ is nice.