Spectral
11/12/09

Embeddings

$G=(V,E)$ valuation $X_1 \cdots X_k$ (eigenvectors)

Natural embedding of $G \to \mathbb{R}^k$

$\rho_k(v) = (X_1(v), \cdots, X_k(v))$

Goal: G 3-connected planar, Mult $\lambda_2(Q)=3$, then $\rho_3(v)$ gives a planar embedding of G on sphere.

<u>Lemma</u> G 3-conn planar, $\text{Mult}(\lambda_2(Q))=3$, F a face $(u,v)\in F$ then $\rho_3(u)$ ind $\rho_3(v)$.

<u>pf</u> by contra.

Assume $\rho_3(u) = \alpha \rho_3(v)$ & vertex $w \neq u,v$ $w \in F$.

$$\begin{array}{c|ccc} & X_1 & X_2 & X_3 \\ \hline v_1 & \alpha_1 & \beta_1 & \gamma_1 \\ \vdots & & & \\ v_n & \alpha_n & & \end{array} \Rightarrow \begin{array}{c|ccc} u & \alpha_1 & \beta_1 & \gamma_1 \\ v & \alpha\alpha_1 & \alpha\beta_1 & \alpha\gamma_1 \\ w & \alpha_3 & \beta_3 & \gamma_3 \\ & & & \\ & & & \end{array} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ * \\ \vdots \\ * \end{pmatrix} \equiv \bar{X}$$

$\leftarrow$ eigenvector

$\bar{X}$ zero on 3 vertices of a face! contra!

Thus $\rho(v) \neq 0 \ \forall v$. Thus

$\hat{\rho}(u) = \|\rho(u)\|^{-1}\rho(u)$ is well defined

$\forall (u,v) \in E \Rightarrow \hat{\rho}(u) = \pm\hat{\rho}(v)$

Define $\hat{\rho}((u,v))$ = geodesic from $\hat{\rho}(u)$ to $\hat{\rho}(v)$.

Thus $\hat{\rho} : G \to$ unit sphere

Goal: Show $\hat{\rho}$ gives a planar embedding.

Lemma G 3-conn embedded planar graph, $\langle v_1 \cdots v_k \rangle$ are neig(u) in their cyclic order then $\langle v_1, \cdots, v_k \rangle$ or $\langle v_k, \cdots, v_1 \rangle$ is their order on sphere.

pf Note given $\rho(G)$ we can apply any rigid rotation to $\rho(G)$ and the x, y, z coordinates are eigenvector.

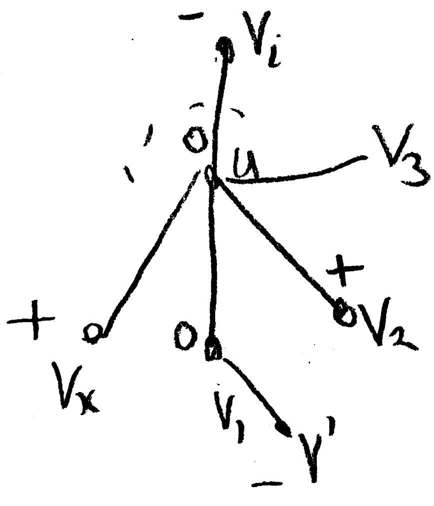
Thus we may assume that $x(u) = x(v_1) = 0$.

By lemma $x(v_2)$ & $x(v_k) \neq 0$.

After a reflection assume $x(v_2) > 0$.

<u>Claim</u> $x(v_k) < 0$ Suppose $x(v_k) > 0$

picture



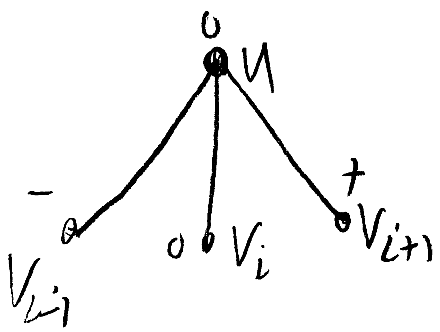
The cycles $(u, v_2, \cdots v_k, u) = C_+$

$(u, v_i, \cdots, v', v_1, u) = C_-$

are disjoint except at u.

And the interlace! contra!

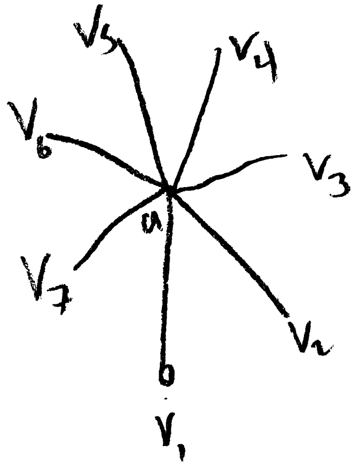
<u>Claim</u> $\Rightarrow \forall v_i$



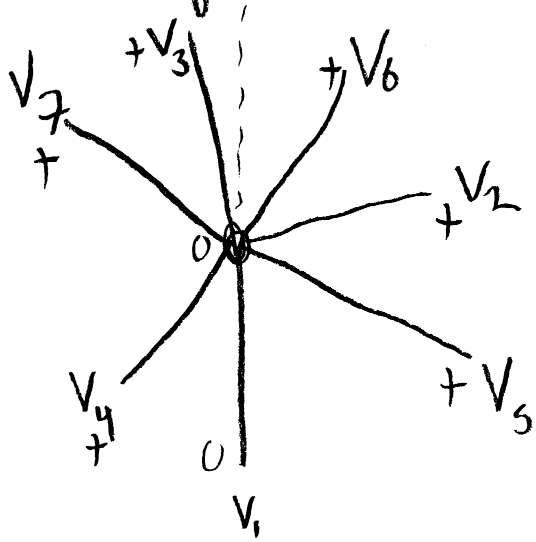
Are we done?

consider example

on plane



on sphere



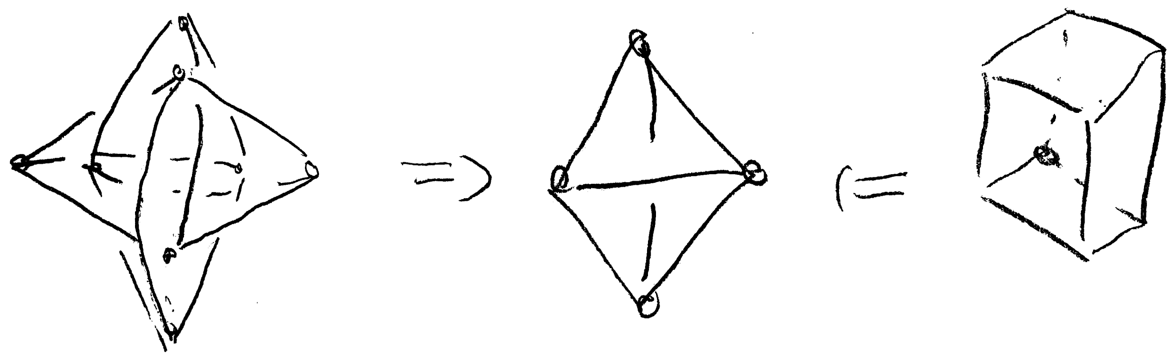
Note sphere consistant with claim

Showing the map $\hat{\rho}$ is injective.

Prob: We may be getting a multi-fold cover!

eg a 2-fold cover of K_4

Does not preserve face size!



For planar covers we can determine the fold using Euler's formula.

$2 = v - e + f$

map: $S_a \longrightarrow S_b$

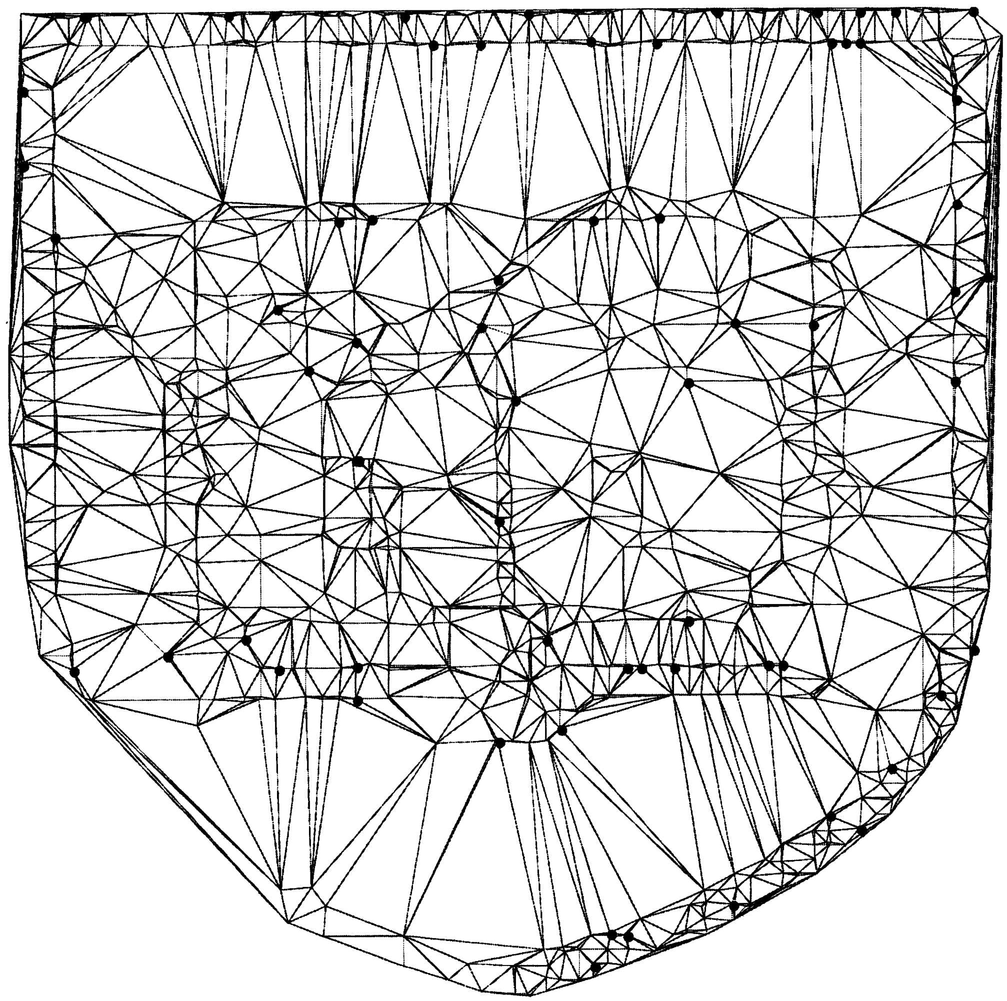
planar embedding $\hat{\rho}$ embedding

$2k = kv - ke + kf$ $2 = v - e + f$

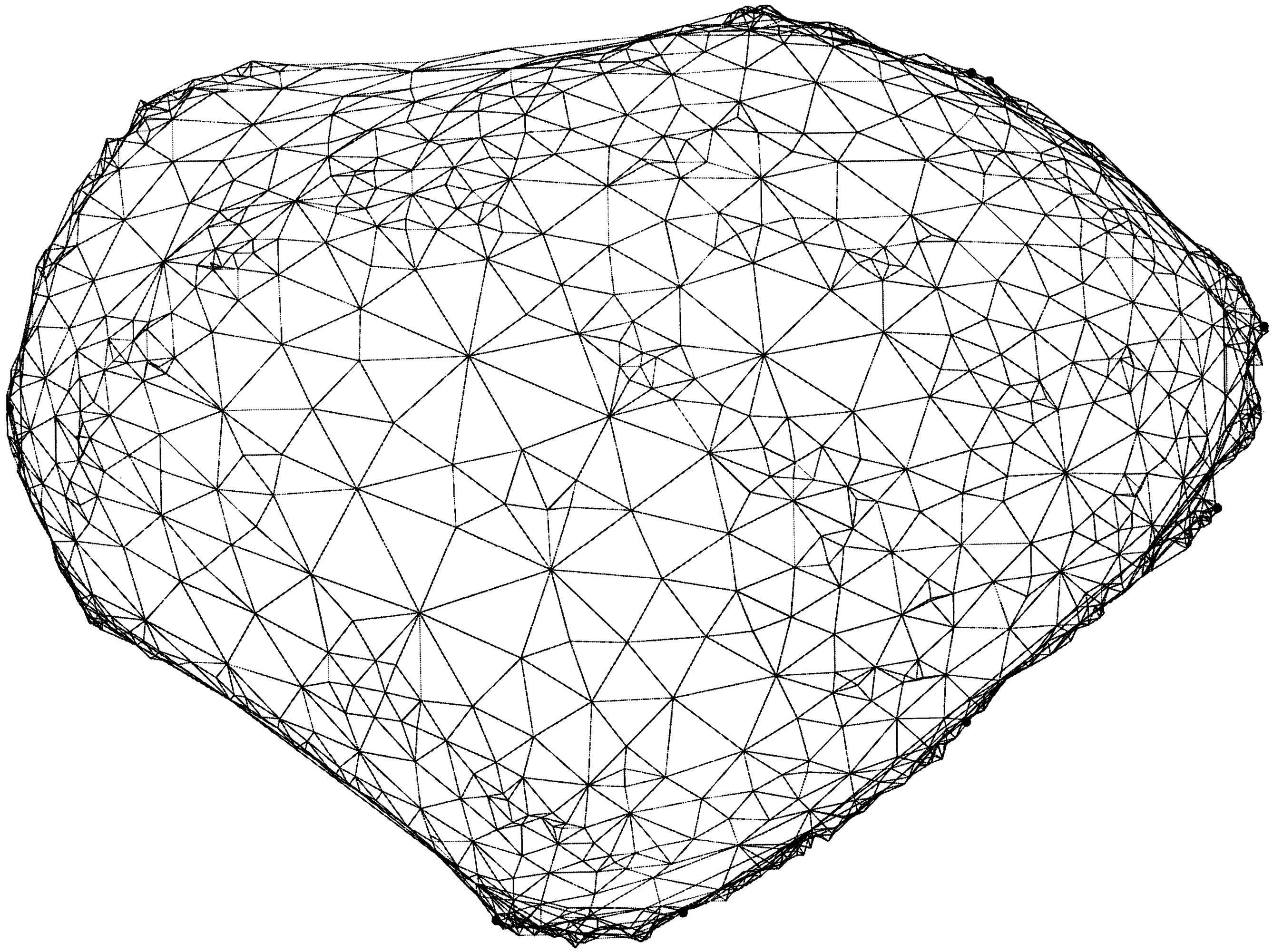
$2k = k(v - e + f)$

$\Rightarrow k = 1$

A Graph



Drawing of the graph using v_2, v_3



Plot vertex i at $(v_2(i), v_3(i))$