Support Tree Preconditioners

Idea: Use a Steinitz tree preconditioner (Support tree)

Example: Input vnx vn mesh Mn

Set edge weights to cut size

2 issues

1) What is our iterative method?
2) How do we estimate $K(A, B)$?
Consider Richardson's

\[ x^{(m+1)} = x^{(m)} + B^{-1}(b - Ax^{(m)}) \]

1) \( R = b - Ax^{(n)} \)

2) \[
\begin{array}{c|c}
\text{new var} & \text{old var} \\
\hline
B_1 & C \\
C^T & B_2 \\
\end{array}
\]

\( \equiv B \)

solve \( Bx = y \)

ie let voltage at \( y \) "float".

3) return \( x^{(n+1)} = x^{(n)} + x \)

Note Since \( B \) is still the Laplacian of a tree step 2) is still \( O(n) \) work.
Question:
Defining & bounding $K(A,B)$

Consider pivoting out the Steiner variables.

$$B(x) = \begin{pmatrix} 0 \\ x \end{pmatrix} \Rightarrow \begin{pmatrix} B_1 & C \\ 0 & B_2 - C^T B_1 C \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} d \\ r \end{pmatrix}$$

$D = B_2 - C^T B_1 C$ (Shur Complement)

$x$ satisfies $DX = r$

Definition $K(A,B) = K(A,D)$

$\Gamma(A/B) = \Gamma(A/D)$ & $\Gamma(B/A) = \Gamma(D/A)$

Estimate $\Gamma(M/T)$

1D case first.

Congestion $\leq 1$

Dilation $\leq \log n$

$\Gamma(P_n/T) \leq \log n$
Quad tree case

Demand on edge \( e \) will be edges leaving \( Q \).

But \( w(e) = \text{Demand} \)

Congestion = \( 1 \)

Dilation \( \leq \log n \)

Claim: Shur Complement is a graph Laplacian.

\( P^F \): Induction on \( \# \) of variables removed.

Consider pivoting on \( V \) neighbors \( V_1, \ldots, V_k \)
edge weights \( w_1, \ldots, w_k \)

\[ d(V) = \sum w_i \]
Check that row sums still zero!

Bounding \( T(T/M) \):

In case and all edge in \( T \) have weight 1.

Consider all edges from left subtree to right subtree

By symmetry they all have the same weight.

Consider 2 electrical experiments

1) In \( T \): setting left leaves to zero & right leaves to 1

2) In \( D \): "node" "node" "node"

Net flow should be the same.
\[ \text{In } T \]

\[ \text{Conductors} \]

\[ R_0 = 1 \quad R_1 = \frac{1}{2} \quad R_2 = \frac{1}{4} \quad \ldots \]

\[ 2 \sum R_i = \sum \frac{1}{2^i} = O(1) \]

\[ \Rightarrow \text{ Current } \in \Theta(1) \]

\[ \text{In } D \quad \binom{n}{2} \text{ edges} \]

\[ \Rightarrow \text{ each edge weight } \approx \frac{1}{n^2} \]
Edup from left to right are subgraph \( \frac{1}{n^2} k_{n/2} n/2 = B \)

\[ \nabla \left( \frac{1}{n^2} k_{n/2} n/2 \right) \leq C \cdot D \]

\( C = 1, \quad D = n \implies \gamma \leq n \)

Consider edge one "level up" \( B_2 \)

\[ \Rightarrow \nabla(B_2, P_n) \leq \frac{n}{2} \]

\[ \nabla(B/P_n) = \nabla \left( \sum B_i / P_n \right) = \sum \nabla(B_i / P_n) \]

\[ = \sum \frac{n}{2^i} = O(n) \]

\[ k(B, P_n) = O(n \log n) \]

Claim \( K(Q_n, M_n) = O((\sqrt{n} \log n)) \) 2D case

\( Q_n \): quad tree with \( n \) leaves

\( M_n \): \( \sqrt{n} \times \sqrt{n} \) mesh.
Bounding $\Upsilon(T/G)$ using Power.

**Theorem**

If $G, H$ are graphs such that $V(G) \subseteq V(H)$, then

$\Upsilon(H/G) = \min \{ \tau \mid \forall x \exists y \quad \tau x^T L_G x \geq (y)^T L_H (y) \}$

**Proof**

Let $L^*_H$ be the Schur complement of $L_H \cap V(G)$

$\Upsilon(H/G) = \Upsilon(L^*_H / L_G) = \min \{ \tau \mid \forall x \quad \tau x^T L_G x \geq x^T L^*_H x \}$

Set $y$ s.t. $L_H (y) = (y)$

Then

$X^T L^*_H X = (y)^T L_H (y)$
Example
Suppose our tree is

Let's estimate $k(F_n, P_n)$.

$\mathcal{O}(p_n / F_n) \leq C \cdot D = 2.3 \leq 6$

Claim: $\mathcal{O}(F_n / p_n) \leq 1 \quad \ell^* = \text{char complement}$

Goal: Given $X$ bd $\frac{x^T \ell^* x}{x^T p_n x}$

By Thm we can pick any $y$ to get an upper bd.

Set $y$ s.t.

\[ x_1 \quad x_2 \quad \cdots \quad x_n \]

\[ x_1 \quad x_2 \quad \cdots \quad x_n \]

$\leq (y)^T L^* (y) = x^T p_n x$
Example 2  Redo binary tree for path.

\[ \begin{align*}
\text{Set } y \text{ to the average.} \\
\frac{v_1 + v_4}{2} & \quad v_1 \quad v_4 \\
\frac{v_1 + v_2}{2} & \quad v_2 \\
X = v_1 & \quad v_n
\end{align*} \]

Question  What is power consumed by setting \((X)\)?

**Thm** \( S = \sum_{i=1}^{n} \alpha_i = \sum_{j=1}^{n} \beta_j \) \& \( \alpha_i, \beta_j \geq 0 \) then

\( \forall v, w \quad \left( \sum \alpha_i v_i - \sum \beta_j w_j \right)^2 \leq \sum \alpha_i \beta_j (v_i - w_j)^2 \)

**Proof sketch:**

\[ \text{RHS - LHS} = \sum \alpha_i \alpha_j (v_i - v_j)^2 + \sum \beta_i \beta_j (w_i - w_j)^2 \]

**1 ≤ i, j ≤ n**
1) To bd $\nabla(T_n/2_n)$ we bd power in each level of $T_n$

2) Consider top two edges of $T_n$

Replace with a single edge with weight $m$

$$\begin{align*}
\begin{array}{c}
\text{ replaced by }
\end{array}
\end{align*}$$

$$\frac{V_i + \cdots + V_n}{n} \cdot \frac{2^H}{n} \Rightarrow V_i^H \cdots V_n^H$$

Power on $e_i$:

$$\left( \frac{2^H}{n} \frac{a_n}{n} V_i - \sum_{i=1}^{n} \frac{2^H}{n} V_i^H \right)^2 \leq \sum_{1 \leq i \leq n} \frac{a_n^2}{n^2} (V_i - V_j)^2$$

RHS is power in $K_{n^2/n^2}$.

By Cong. Dil

$$\gamma(K_{n^2/n^2} / \rho_n) \leq n^2 \cdot n \leq n^3$$
Power on top level edges \( \leq n V^T P_n V \)
next four edges \( \leq \frac{b}{a} V^T P_n V \)

Total \( \leq O(n) V^T P_n V \)

Thus \( T(T_n / p_n) = O(n) \)