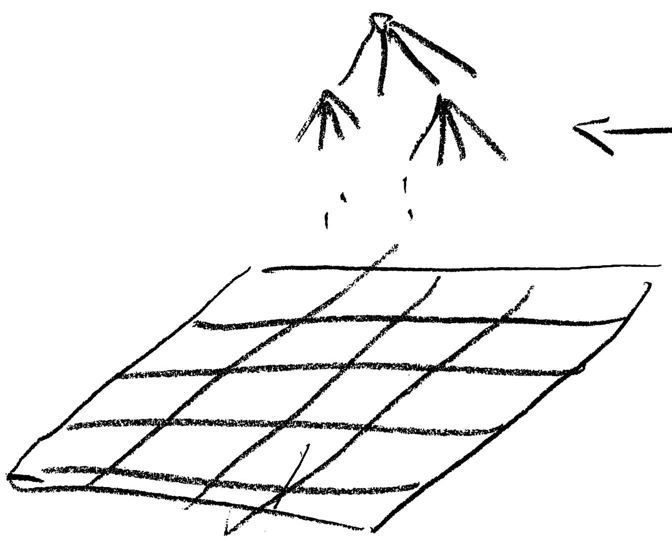
Spectral
11/5/09

Support Tree Preconditioners

Idea: Use a Steiner tree preconditioner
(Support tree)

eg Input $\sqrt{n} \times \sqrt{n}$ mesh M_n



Quad tree

Set edge weights to cut size.

2 issues

1) What is our iterative method?

2) How do we estimate $\kappa(A,B)$?

Consider Richardson's

$$x^{(n+1)} = x^{(n)} + B^{-1}(b - Ax^{(n)})$$

1) $r = b - Ax^{(n)}$

2)
$$\begin{array}{c|c|c|} & \text{new var} & \text{old var} \\ \hline \text{new var} & B_1 & C \\ \hline \text{old var} & C^T & B_2 \\ \hline \end{array} \equiv B$$

solve $B\begin{pmatrix} y \\ X \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}$ ie. let voltage at y "float".

3) return $x^{(n+1)} = x^{(n)} + X$

<u>Note</u> Since B is still the Laplacian of a tree step 2) is still $O(n)$ work.

Question:

Defining & bounding $K(A,B)$

Consider pivoting-out the Steiner variables.

$$B\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix} \Rightarrow \left(\begin{array}{c|c} B_1 & C \\ \hline 0 & B_2 - C^T B_1^{-1} C \end{array}\right)\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}$$

$D = B_2 - C^T B_1^{-1} C$ (Shur Complement)

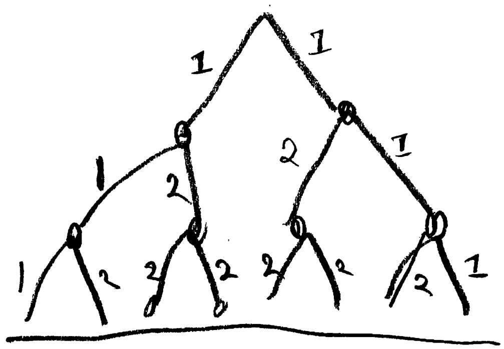
x satisfies $Dx = r$

<u>Definition</u> $K(A,B) = K(A,D)$

$\sigma(A/B) = \sigma(A/D)$ & $\sigma(B/A) = \sigma(D/A)$

Estimate $\sigma(M/T)$

1D case first.

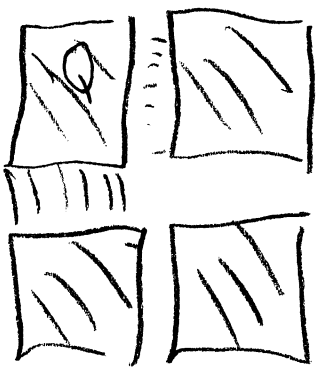


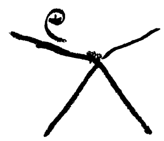
Congestion ≤ 1

Dilation $\leq \log n$

$\sigma(P_n/T) \leq \log n$

Quad tree case





Demand on edge e will be the edges leaving Q.

But w(e) = Demand

Congestion = 1

Dilation $\le \log n$

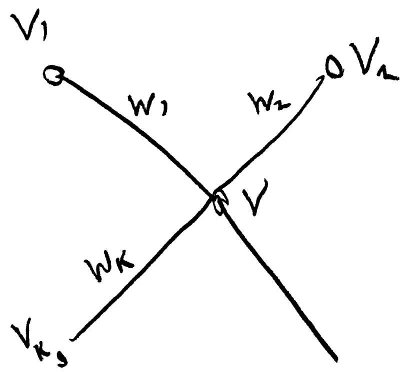
Claim Shur Complement is a graph Laplacian.

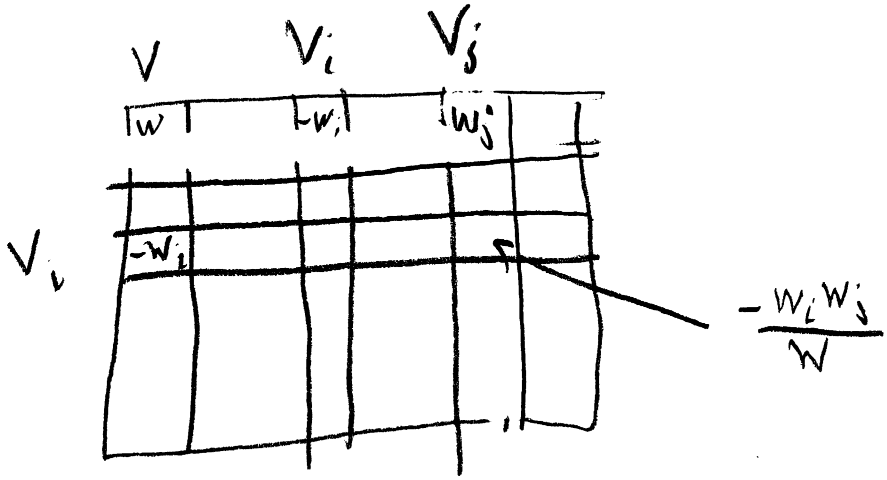
pf Induction on # a variables removed.

Consider pivoting on V neighbors $V_1 \ldots V_k$

edge weights $w_1 \ldots w_k$

$d(V) = \sum w_i$





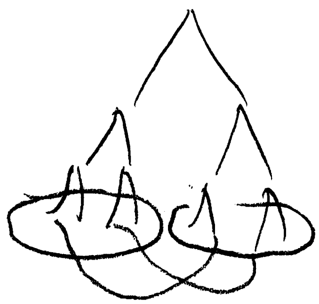
Chech that row sums still zero!

Bounding $\sigma(T/M)$:

<u>1D case</u> and all edge in T have weight 1.

Consider all edges from left subtree to right subtree in D.

is



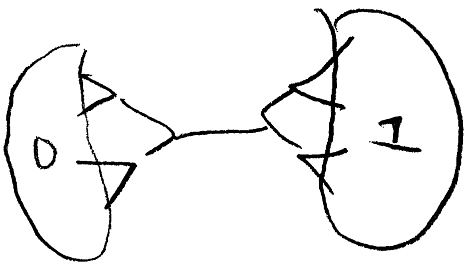
By symmetry they all have the same weight.

Consider 2 electrical experiments

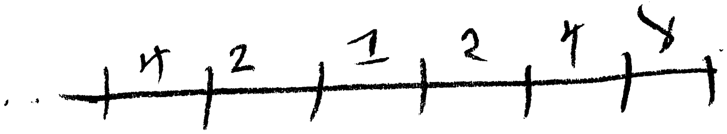
1) In T: Setting left leaves to zero & right leaves to 1
2) In D: " "nodes" " "nodes" "!

Net flow should be the same.

In T



conductors

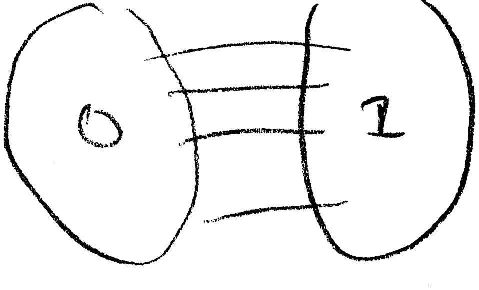


Resistor $R_0 = 1$ $R_1 = 1/2$ $R_0 = 1/4$, ...

$$2\sum R_i = \sum \frac{1}{2^i} = O(1)$$

$\therefore$ Current is $\Theta(1)$

In D $\binom{n}{2}$ edges



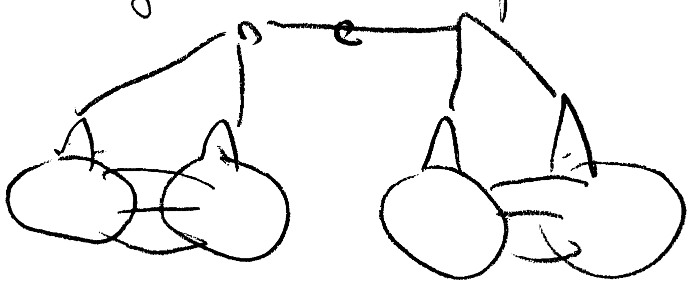
$\therefore$ each edge weight $\approx \frac{1}{n^2}$

Edges from left to right are subgraph $\frac{1}{n^2} K_{n/2\, n/2} = B_1$

$$\nabla\left(\tfrac{1}{n^2}K_{n/2\,n/2} \,/\, P_n\right) \le C \cdot D$$

$$C = 1,\ D \le n \implies \nabla \le n$$

Consider edge one "level up" B_2



$$\Rightarrow \nabla(B_2, P_n) \le n/2$$

$$\nabla(B/P_n) = \nabla\left(\sum B_i / P_n\right) = \sum \nabla(B_i / P_n)$$

$$= \sum n/2^i = O(n)$$

$$K(B, P_n) = O(n \log n)$$

<u>Claim</u> $K(Q_n, M_n) = O(\sqrt{n} \log n)$ 2D. case

Q_n: quad tree with n leaves

M_n: $\sqrt{n} \times \sqrt{n}$ mesh.

Bounding $\tau(T/G)$ using Power.

Thm G, H graph s.t. $V(G) \subseteq V(H)$ then

$$\tau(H/G) = \min \left\{ \tau \mid \forall x \, \exists y \;\; \tau x^T L_G x \geq \begin{pmatrix} y \\ x \end{pmatrix}^T L_H \begin{pmatrix} y \\ x \end{pmatrix} \right\}$$

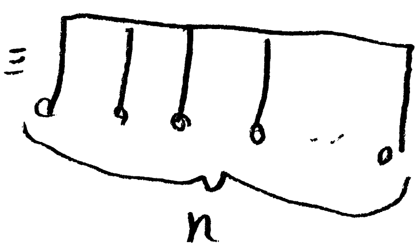
Pf Let $L_H^* \equiv$ Shur complement of $L_H \restriction V(G)$

$$\tau(H/G) = \tau\left(L_H^* / L_G\right) = \min \left\{ \tau \mid \forall x \;\; \tau x^T L_G x \geq x^T L_H^* x \right\}$$

set y s.t. $L_H \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ ? \end{pmatrix}$

$$x^T L_H^* x = \begin{pmatrix} y \\ x \end{pmatrix}^T L_H \begin{pmatrix} y \\ x \end{pmatrix}$$

__Example__ Suppose our tree is $F_n \equiv$



Lets estimate $k(F_n, P_n)$.

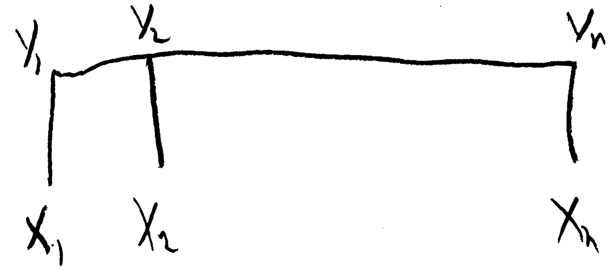
$\sigma(P_n/F_n) \le C \cdot D = 2 \cdot 3 = 6$

Claim: $\sigma(F_n/P_n) \le 1$ $\quad L_F^* =$ schur complement

Goal: Given x bd $\dfrac{x^T L_F^* x}{x^T P_n x}$.

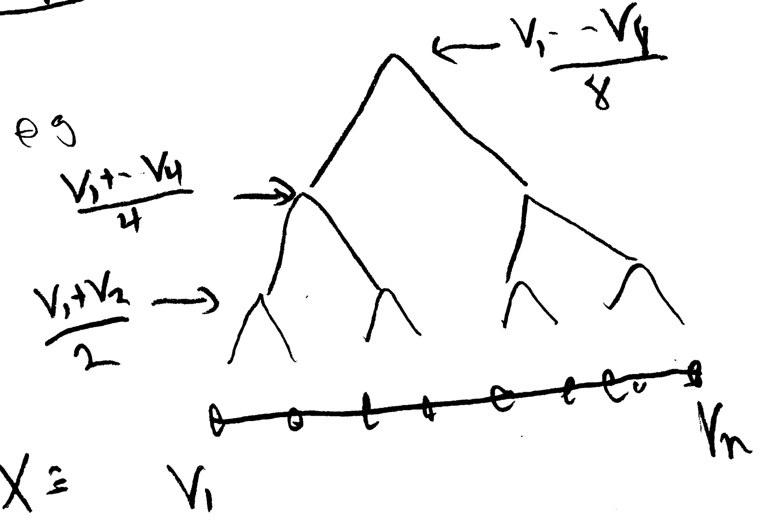
By Thm we can pick any y to get an upper bd.

Set y s.t



$\le \begin{pmatrix} y \\ x \end{pmatrix}^T L_F \begin{pmatrix} y \\ x \end{pmatrix} = x^T P_n x$

__Example 2__ Redo binary tree for path.



Set y to the average.

__Question__ What is power consumed by setting $\binom{y}{x}$?

__Thm__ $\delta = \sum_{i=1}^{m} \alpha_i = \sum_{j=1}^{n} \beta_j$ & $\alpha_i, \beta_j \geq 0$ then

$$\forall v, w \quad \left(\sum \alpha_i v_i - \sum \beta_j w_j \right)^2 \leq \sum \alpha_i \beta_j (v_i - w_j)^2$$

__pf__ sketch!

$$\text{RHS} - \text{LHS} = \sum_{1 \leq i < j \leq m} \alpha_i \alpha_j (v_i - v_j)^2 + \sum_{1 \leq i < j \leq n} \beta_i \beta_j (w_i - w_j)^2$$

1) To bd $\sigma(T_n/L_n)$ we bd power in each level of T_n

2) Consider top two edges of T_n

Replace with a single edge with weight $1/2$

$$\frac{V_1+\cdots V_{n/2}}{n/2} \qquad \frac{V_{n/2+1}+\cdots V_n}{n/2}$$

Power on e:

$$\left(\sum_{i=1}^{n/2}\frac{2}{n}V_i-\sum_{j=n/2+1}^{n}\frac{2}{n}V_j\right)^2 \le \sum_{\substack{1\le i\le n/2\\ \frac{n}{2}+1\le j\le n}}\frac{4}{n^2}(V_i-V_j)^2$$

RHS is power in $K_{n/2\,n/2}$.

By Cong. Dil

$$\sigma(K_{n/2\,n/2}/P_n)\le n^2\cdot n \approx n^3$$

Power on top level edges $\leq n \, v^T P_n v$

next four edges $\leq \frac{n}{2} v^T P_n v$

$\vdots$

Total $\leq O(n) \, v^T P_n v$

Thus $\sigma(T_n / P_n) = O(n)$