Spectral
10/22/09
V2

The Basic Iterative Method

Goal: Solve $Au = b$ $(*)$

Let $u^{(0)}$ our initial guess eg $u^{(0)} = 0$

Richardson's Method

view 1 $$u^{(m+1)} = u^{(m)} + (b - Au^{(m)})$$

residual error

view 2 $$u^{(m+1)} = (I - A)u^{(m)} + b$$

$$u^{(m+1)} = Gu^{(m)} + b \qquad G = (I - A)$$

Good: 1) $O(w(A))$ work per step. (parallel)

2) A fixed point is a solution

Bad: 1) May not converge.

2) If it does it may do so very slowly

The error for Richardson

Suppose $A\bar{u}=b$ & $G=(I-A)$

Def the error $\varepsilon^{(m)} = u^{(m)} - \bar{u}$

Claim $\varepsilon^{(m)} = G^{m}\varepsilon^{(0)}$

to show $\varepsilon^{(m+1)} = G\varepsilon^{(m)}$

Pf
$$
\begin{aligned}
G\varepsilon^{(m)} &= G(u^{(m)} - \bar{u}) = Gu^{(m)} - (I-A)\bar{u} \\
&= Gu^{(m)} - \bar{u} + A\bar{u} \\
&= Gu^{(m)} - \bar{u} + b \\
&= (Gu^{(m)} + b) - \bar{u} \\
&= u^{(m+1)} - \bar{u} = \varepsilon^{(m+1)}
\end{aligned}
$$

To converge $\lim_{m \to \infty} G^m \varepsilon^{(0)} = 0$

If it is to converge $\forall \varepsilon^{(0)}$ then $\lim_{m \to \infty} G^m = 0$

If A is sym then G is sym

$$G = \sum \lambda_i (v_i v_i^T) \quad \lambda_i \in \lambda(G) \quad v_i \text{ eigen vector}$$

Iff $\forall_i \; |\lambda_i| < 1$ Spectral radius

In general this is not true!

A simple modification to get convergence.

The Extrapolated Method

Goal $Ax=b$ iff $\gamma Ax = \gamma b$

new recurance $u^{(m+1)} = u^{(m)} + \gamma(b - Ax)$

or $u^{(m+1)} = (I - \gamma A)u^{(m)} + \gamma b$

Let M_A = max eigenvalue of A.

Let m_A = min " " .

Set $G_\gamma = (I - \gamma A)$ $\quad \gamma = \frac{2}{M_A + m_A}$

$$M_{G_\gamma} = 1 - \left(\frac{2}{M_A + m_A}\right) m_A = \frac{M_A + m_A - 2m_A}{M_A + m_A} = \frac{M_A - m_A}{M_A + m_A}$$

$$m_{G_\gamma} = 1 - \frac{2}{(M_A + m_A)} M_A = \frac{m_A - M_A}{M_A + m_A} = -M_{G_\gamma}$$

$$\sigma(G_\gamma) = \frac{M_A - m_A}{M_A + m_A}$$

A spd.

Def condition number of $A \equiv \frac{M_A}{m_A} = k(A)$

$$\sigma(G_\gamma) = \frac{k(A)-1}{k(A)+1} \qquad k = k(A)$$

$$1 - \frac{2}{k+1} = 1 - \frac{1}{\frac{k+1}{2}} \approx 1 - \frac{1}{k}$$

If we do k iterations. Error $\approx \left(1-\frac{1}{k}\right)^k \approx \frac{1}{e}$

We need k iterations per bit of accuracy.

EG $\quad A = L(P_n) \quad k(A) \approx n^2$

n^2 iterations per bit.

Polynomial Acceleration

$$Ax = b \quad \text{spd} \quad A\bar{x} = b$$

$$x^{(m+1)} = Gx^{(m)} + b \qquad G = (I - A)$$

$$\varepsilon^{(m)} = G^m \varepsilon^{(0)}$$

Idea: Use all previous $x^{(i)}$.

Compute $x^{(0)}, \dots, x^{(n)}$

pick $\alpha_i \in \mathbb{R}$ s.t. $\sum \alpha_i = 1$

$$u^{(n)} = \sum \alpha_i x^{(i)}$$

Question: How to pick the α_i

$$\tilde{\varepsilon}^{(n)} = u^{(n)} - \bar{x}$$

$$= \sum_{i=0}^{n-1} \alpha_i x^{(i)} - \bar{x} \qquad \sum \alpha_i = 1$$

$$= \sum \alpha_i (x^{(i)} - \bar{x})$$

$$= \sum \alpha_i \varepsilon^{(i)} = \sum \alpha_i G^i \varepsilon^{(0)}$$

$$= \left(\sum \alpha_i G^i\right) \tilde{\varepsilon}^{(0)} = Q_n(G)\, \tilde{\varepsilon}^{(0)}$$

$$Q_n(z) = \alpha_0 + \alpha_1 z + \cdots \alpha_{n-1} z^{n-1} \qquad \sum \alpha_i = 1$$

Goal: For each n pick $Q_n(z)$ to

min $\|Q_n(G)\|$ u min $\sigma(Q_n(G))$

Chebyshev Acceleration

The Chebyshev Polys

$$T_0(w) = 1 \qquad T_1(w) = w$$

$$T_{n+1}(w) = 2w\,T_n(w) - T_{n-1}(w)$$

eg $T_2(w) = 2w \cdot w - 1 = 2w^2 - 1$

$$T_3(w) = 2w(2w^2 - 1) - w$$

$$= 4w^3 - 2w - w = 4w^3 - 3w$$

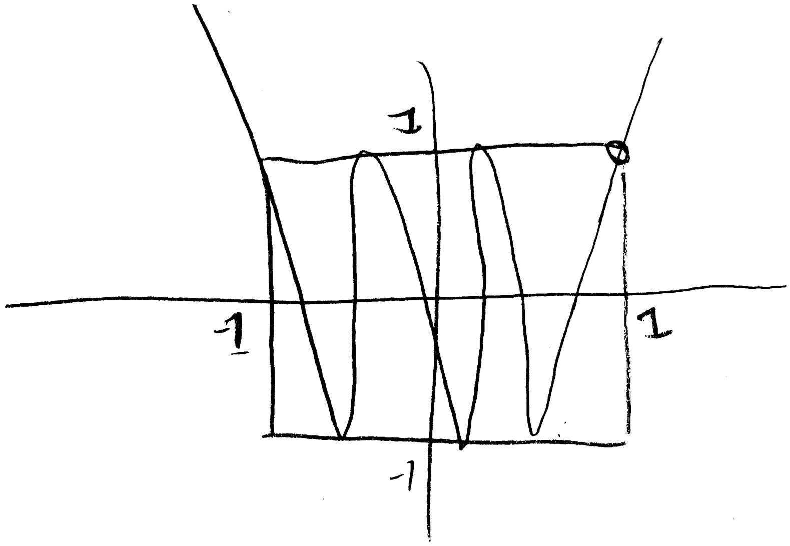
<u>Note</u> $T_n(1) = 1 \Rightarrow \sum \text{coef} = 1$

$$T_n(-1) = (-1)^n$$

$$T_n(w) = \frac{1}{2}\left[(w + \sqrt{w^2 - 1})^n + (w + \sqrt{w^2 - 1})^{-n}\right]$$

HW!

T_n even



<u>Thm</u> $d > 1$ Let $H_n(w) = T_n(w)/T_n(d)$ then

1) $\max_{-1 \le w \le 1} |H_n(w)| = 1/T_n(d)$

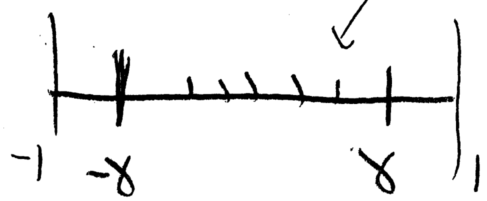
2) $H_n(w)$ is min such poly

Chebyshev & extrapolated method.

$\kappa(A) = \kappa$ & $\gamma = \frac{\kappa - 1}{\kappa + 1}$

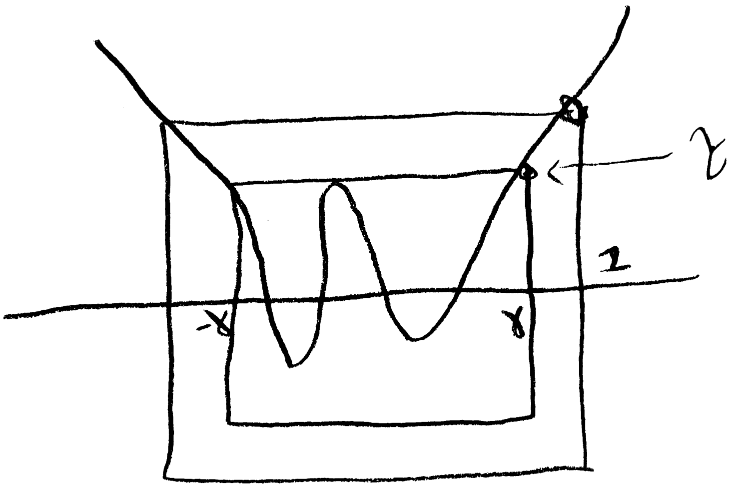
eigenvalues of G.

$\lambda(G)$



Pick $\bar{T}_n = T_n(x/\gamma) / T_n(1/\gamma)$

$\bar{T}_n(1) = 1 \quad \sum \text{coef} = 1$



Using Thm: $1 = x/\gamma \Rightarrow x = \gamma$

$\tau = 1 / T_n(1/\gamma)$

<u>Goal</u> bd $T_n(1/\gamma)$

$$1/\gamma = \frac{K+1}{K-1} = 1 + \frac{2}{K-1} = 1 + 2\mu \quad \text{if } \mu = \frac{1}{K-1}$$

$$T_n(x) = \frac{1}{2}\left[\left(x + \sqrt{x^2-1}\right)^n + \left(x + \sqrt{x^2-1}\right)^{-n}\right]$$

$$\geq \frac{1}{2}\left(x + \sqrt{x^2-1}\right)^n \quad x \geq 1$$

$$T_n(1+2\mu) \geq \frac{1}{2}\left(1 + 2\mu + \sqrt{(1+2\mu)^2 - 1}\right)^n$$

$$= \frac{1}{2}\left(1 + 2\mu + 2\sqrt{\mu(\mu+1)}\right)^n$$

$$= \frac{1}{2}\left(\sqrt{\mu} + \sqrt{\mu+1}\right)^{2\mu} \quad (*)$$

$$\left(\sqrt{\mu} + \sqrt{\mu+1}\right)^2 = \left(\frac{1}{\sqrt{K-1}} + \frac{\sqrt{K}}{\sqrt{K-1}}\right)^2 = \frac{(1+\sqrt{K})^2}{K-1} = \frac{\sqrt{K}+1}{\sqrt{K}-1}$$

$$(*) = \frac{1}{2}\left(\frac{\sqrt{K}+1}{\sqrt{K}-1}\right)^n$$

$$T_n(1/\gamma) \geq \frac{1}{2}\left(\frac{\sqrt{\kappa}+1}{\sqrt{\kappa}-1}\right)^n$$

$$\gamma \leq 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^n$$

Thm Convergence Rate for Extrapolated Chebyshev is $\Theta(1/\sqrt{\kappa})$

Back to P_n $\quad \kappa(L(P_n)) = n^2$

$\therefore$ Convergence Rate for Chebyshev is

$\Theta(1/n)$

ie n iterations per bit.

From Polynomial Recurance to
Iterative Alg

Consider a poly defined by

$$Q_0(x) = 1$$

$$Q_1(x) = \alpha_1 x + \beta_1 \qquad \alpha_1 + \beta_1 = 1$$

$$Q_{n+1}(x) = \alpha_n x\, Q_n(x) + \beta_n Q_{n-1}(x)$$

Iterative Alg

$$u^{(n+1)} = \alpha_n\left(G u^{(n)} + b\right) + \beta_n u^{(n-1)}$$

<u>to show</u> $\varepsilon^{(n)} = u^{(n)} - \bar{u}$ then $\varepsilon^{(n)} = Q_n(G)\,\varepsilon^{(0)}$

$$Q_{n+1}(G)\,\varepsilon^{(0)} = \left[\alpha_n G\, Q_n(G) + \beta_n Q_{n-1}(G)\right]\varepsilon^{(0)}$$

$$= \alpha_n G \underbrace{Q_n(G)\,\varepsilon^{(0)}} + \beta_n Q_{n-1}(G)\,\varepsilon^{(0)}$$

$$= \alpha_n G\left(\varepsilon^{(n)}\right) + \beta_n\left(\varepsilon^{(n-1)}\right)$$

$$= \alpha_n G(u^{(n)} - \bar{u}) + \beta_n (u^{(n-1)} - \bar{u})$$

$$= \alpha_n (G u^{(n)} - \bar{u} + b) + \beta_n u^{(n-1)} - \beta \bar{u}$$

$$= \alpha_n (G u^{(n)} + b) + \beta_n u^{(n-1)} - \bar{u}$$

$$= u^{(n+1)} - \bar{u} = \varepsilon^{(n+1)}$$