

Spectral  
10/15/09  
v2

# Graph Cuts & Eigenvalues

2-types of cuts

edge-cuts & vertex-cuts

Today edge-cuts

$G = (V, E, w)$  undirected weighted graph

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Def  $\text{Cap}(A, B) = \sum_{\substack{(a,b) \in E \\ a \in A, b \in B}} w(a, b)$

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Cut measures

1) Quotient cuts or Isoperimetric number

$$\delta(G) = \min_{S \subseteq V} \frac{\text{Cap}(S, \bar{S})}{\min\{|S|, |\bar{S}|\}}$$

2) Sparses Cut

$$\alpha(G) = \min_{S \subseteq V} \frac{\text{Cap}(S, \bar{S})}{|S| |\bar{S}|}$$


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Note  $\frac{n}{2} \alpha(G) \leq \beta(G) \leq n \alpha(G)$

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## 3) Conductance

$$\bar{\rho}(G) = \min_{S \subseteq V} \frac{\text{Cap}(S, \bar{S})}{\min\{\text{Cap}(S), \text{Cap}(\bar{S})\}}$$


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## 4) Sparsest Cut

$$\bar{\lambda}(G) = \min_{S \subseteq V} \frac{\text{Cap}(S, \bar{S})}{\text{Cap}(S) \cdot \text{Cap}(\bar{S})}$$

another view

$$\text{Demand}(v_i, v_j) = d_i \cdot d_j$$

$$\text{Demand}(S, \bar{S}) = \sum_{\substack{s \in S \\ s' \in \bar{S}}} d_s \cdot d_{s'}$$

$$= \left( \sum_{s \in S} d_s \right) \left( \sum_{s' \in \bar{S}} d_{s'} \right) = \text{Cap}(S) \circ \text{Cap}(\bar{S})$$


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more generally

$$\bar{\lambda}(G) = \min_{S \subseteq V} \frac{\text{Cap}(S, \bar{S})}{\text{Demand}(S, \bar{S})}$$

some demand fcn

eg product demand

$$\text{Demand}(v_i, v_j) = P_i \circ P_j$$


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note  $\Phi(G)$  is  $\approx P_i = \frac{d_i}{\sum d_j}$

## Main Thms

$L \equiv$  Laplacian of  $G$      $\lambda_2 = \lambda_2(L)$

Thm 1     $\lambda_2/2 \leq \rho(G) \leq \sqrt{2\Delta} \lambda_2$

$$\Delta \equiv \max_i \{d_i\}$$

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$\mathcal{L} \equiv$  Normalized Laplacian of  $G$      $\lambda_2 = \lambda_2(\mathcal{L})$

ie  $\mathcal{L} = D^{-1/2} L D^{-1/2}$

Thm 2     $\lambda_2/2 \leq \overline{\rho}(G) \leq \sqrt{2} \lambda_2$

Claim  $\lambda_2(L) \leq 2 \cdot \varphi(G)$

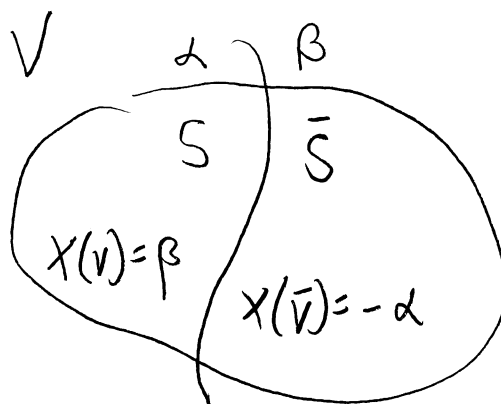
ie  $\exists$  vector  $X$  st  $X^T \mathbf{1} = 0$  &  $\frac{X^T L X}{X^T X} \leq 2 \cdot \varphi$

pf Let  $S \subseteq V$  be set,  $|S| \leq n/2$

$$\frac{\text{Cap}(S, \bar{S})}{|S|} = \varphi$$

set  $\alpha = |S|$  &  $\beta = |\bar{S}|$

Def  $X_i = \begin{cases} \beta & \text{if } i \in S \\ -\alpha & \text{otherwise} \end{cases}$



$$X^T \mathbf{1} = \alpha \cdot \beta - \alpha \beta = 0$$

$$\frac{X^T L X}{X^T X} = \frac{\text{Cap}(S, \bar{S}) (\alpha + \beta)^2}{\alpha \beta^2 + \beta \alpha^2} = \frac{\text{Cap}(S, \bar{S}) n^2}{\alpha \cdot \beta \cdot n} = \left( \frac{\text{Cap}(S, \bar{S})}{\alpha} \right) \left( \frac{n}{\beta} \right)$$

$$\leq \varphi \cdot 2$$

pf of upper bd for Thm 2.

We spring-mass version  $LX = \lambda DX$

Given  $X^T \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = 0$

We show  $\Phi(G) \leq \sqrt{2 \frac{X^T L X}{X^T D X}}$  (\*)

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The proof is effective!

ie given  $X$  we find a "good" cut.

A "threshold cut"

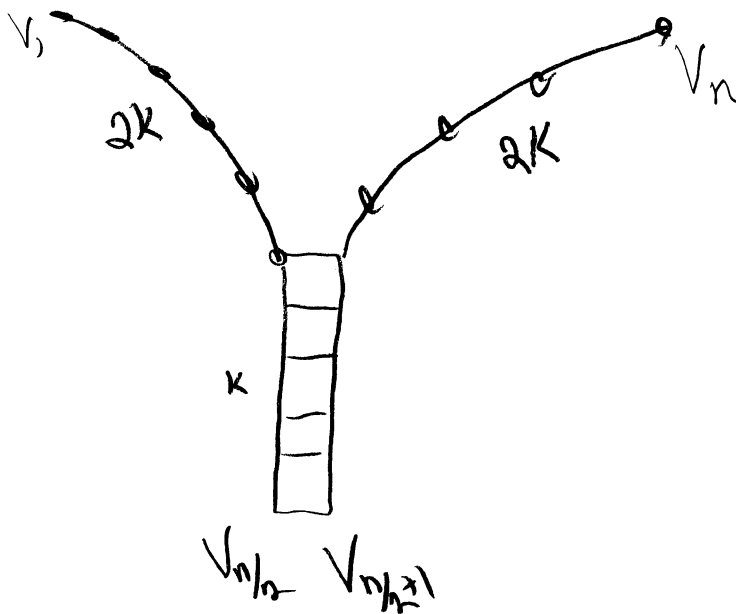
Suppose  $f: V \rightarrow \mathbb{R}$

sort  $V$  so that  $f(v_i) \geq f(v_{i+1})$

Def  $i$ th threshold cut  $\equiv S_i = \{v_1, \dots, v_i\}$

Note  $S_{n/2}$  may not be "good".

Consider Bug-Graph



Claim If  $Lf = \lambda_2 f$  then  $f$  is odd with respect to its reflection.

The threshold cut for  $S_{n/2}$  gives  $\frac{k}{3k} \approx \frac{1}{3}$

$$f \approx \frac{1}{2k} = \frac{3}{n}$$

Preliminaries  $A \equiv$  adj matrix

$$L = D - A \quad z^T L z = \sum_{i,j} w_{ij} (z_i - z_j)^2$$

The sum-Laplacian  $P = D + A$

Claim  $z^T P z = \sum_{i,j} w_{ij} (z_i + z_j)^2$

Fact  $P = 2D - L$

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Cauchy-Schwartz

$$\begin{aligned} (z^T L z)(z^T P z) &= \left( \sum w_{ij} (z_i - z_j)^2 \right) \left( \sum w_{ij} (z_i + z_j)^2 \right) \\ &= \left( \sum (\sqrt{w_{ij}} |z_i - z_j|)^2 \right) \left( \sum (\sqrt{w_{ij}} |z_i + z_j|)^2 \right) \\ &\geq \left( \sum w_{ij} |z_i^2 - z_j^2| \right)^2 \end{aligned}$$



The proof  $z$  st  $z^T D \bar{1} = 0$   $z_1 \geq z_2 \geq \dots \geq z_n$

$$\Phi_i = \frac{\text{Cap}(S_i, \bar{S}_i)}{\min\{\text{Cap}(S_i), \text{Cap}(\bar{S}_i)\}}$$

To show  $\exists i \quad \Phi_i \leq \sqrt{\frac{z^T L z}{z^T D z}}$

We modify  $z$  &  $G$  only decreasing  $\frac{z^T L z}{z^T D z}$  s.t.

$$\beta = \text{argmin}_i \{ \text{Cap}(\bar{S}_i) \geq \text{Cap}(S_i) \}$$

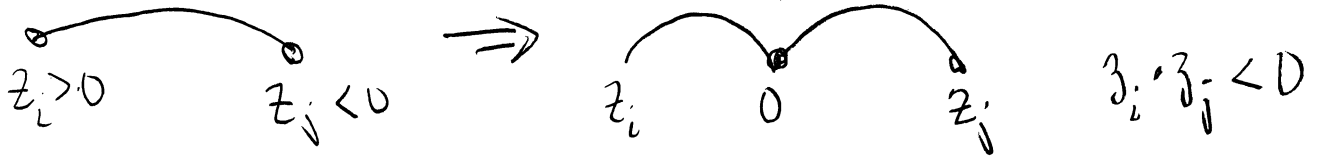
1)  $z_\beta = 0$  (shifting  $z$ )

2)  $\forall (i,j) \in E \quad z_i \cdot z_j \geq 0$  (add new point & set to zero)

1) Shifting  $z$ :  $y = z + \alpha \bar{1}$

$$\frac{y^T L y}{y^T D y} = \frac{(z + \alpha \bar{1})^T L (z + \alpha \bar{1})}{(z + \alpha \bar{1})^T D (z + \alpha \bar{1})} = \frac{z^T L z}{z^T D z + 2\alpha z^T D \bar{1} + \alpha^2 \sum d_i} \leq \frac{z^T L z}{z^T D z}$$

pinching an edge at zero



Claim  $\frac{y^T L' y}{y^T D' y} \leq \frac{z^T L z}{z^T D z}$

Note  $y^T D' y = z^T D z$

To Show  $y^T L' y \leq z^T L z$

term  $(z_i - z_j)^2$  replaced with  $(z_i - 0)^2 + (0 - z_j)^2$

but  $(z_i - z_j)^2 = z_i^2 - 2z_i z_j + z_j^2 > z_i^2 + z_j^2$

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For sum-Laplacians

$$\frac{y^T P' y}{y^T D' y} = \frac{y^T (2D' - L) y}{y^T D' y} \leq \frac{y^T (2D) y}{y^T D' y} = 2$$

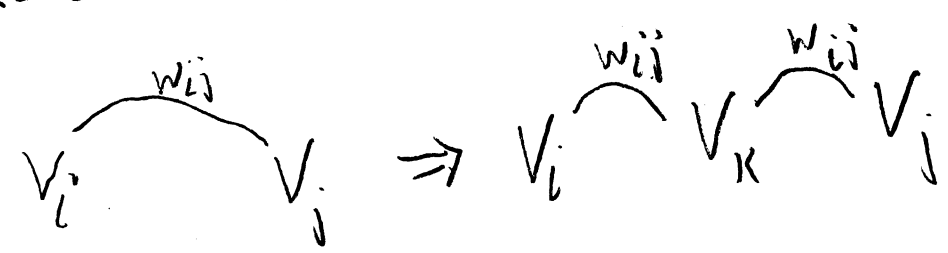

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$$2 \frac{z^T L z}{z^T D z} \cdot (y^T D' y)^2 \geq (y^T P' y)(y^T L' y) \geq \left( \sum_E w_{ij} |y_i^2 - y_j^2| \right)^2$$

$$\sqrt{2 \frac{z^T L z}{z^T D z}} (y^T M y) \geq \sum_{\substack{E \\ i < j \leq \beta}} w_{ij} (y_i^2 - y_j^2) + \sum_{\substack{E \\ j > i \geq \beta}} w_{ij} (y_j^2 - y_i^2) \quad \neq$$

Lets assume  $y_i > y_{i+1}$

WLOG Assume  $G$  is weighted path graph



new edge weights  $w_i$  from  $V_i$  to  $V_{i+1}$

$$\Phi_i = \frac{w_i}{\text{Cap}(V_i)} \quad \Phi^* = \min_{i \in \beta} \Phi_i$$

$$\begin{aligned} \sum_{i=1}^{\beta} w_i (y_i^2 - y_{i+1}^2) &= \sum_{i=1}^{\beta} \Phi_i \text{cap}(V_i) (y_i^2 - y_{i+1}^2) \\ &\geq \Phi^* \sum_{i=1}^{\beta} \text{cap}(V_i) (y_i^2 - y_{i+1}^2) \\ &= \Phi^* \sum_{i=1}^{\beta} (\text{cap}(V_i) - \text{cap}(V_{i-1})) y_i^2 \\ &= \Phi^* \sum_{i=1}^{\beta} w_i y_i^2 \end{aligned}$$

by \*

$$\sqrt{2 \frac{z^T L z}{z^T D z}} (y^T M y) \geq \Phi^* (y^T M y)$$

Thus  $\sqrt{2 \frac{z^T L z}{z^T D z}} \geq \Phi^*$