More Direct Methods

$G = (V, E)$

**Def** \( S \subseteq V \) is an \( f(n) \)-separator if (vertex)

1) \( |S| \leq f(n) \)

2) \( \exists \) partition \( A, B \) of \( V - S \)

3) \( |A|, |B| \leq \frac{2}{3}n \)

**Gilbert-Tarjan** \( S \subseteq V \) is an \( (\alpha, \beta) \)-\( f(n) \)-separator

\( \alpha < 1 \) & \( \beta > 0 \)

1) \( |S| \leq \beta f(n) \)

2) \( \exists \) partition \( A, B \) of \( V - S \)

3) \( |A| \leq \alpha n \)

**Lipton-Tarjan** If \( G \) is planar \( \exists \) \( \sqrt{n} \)-separator

\( \alpha = \frac{2}{3}, \beta = \sqrt{4} \)

**Conjecture** \( \beta = \sqrt{n} \) works.
**Pivot Strategies**

1) *(Nested Dissection) (ND)*
   1) Find a vertex separator $S$ of $G$.
   2) Let $G_1, \ldots, G_k$ be connected components of $G-S$.
   3) Return $\text{ND}(G_1), \ldots, \text{ND}(G_k), S$.

2) *(Lipton-Rose-Tarjan) (Generalized ND)*

Procedure $\text{LRT}(G, H, SV)$

* Returns an ordering of $V-H$.
* Initially $H = \emptyset$.

1) Find vertex separator $S$ of $G$ & part $A, B$.
   a) Let
   
   $G_1 = (A \cup S, E(A \cup S) - E(S))$  $H_1 = SV(H \cap A)$
   $G_2 = (B \cup S, E(B \cup S) - E(S))$  $H_2 = SV(H \cap B)$

   3) Return
   $\text{LRT}(G_1, H_1), \text{LRT}(G_2, H_2), S-H$. 
$\mathcal{S}_2$ be a class of graph closed under subgraphs s.t.
1) Each graph has an $n^d$-separator.

Thin (LRT) for $G \in \mathcal{S}_{1/2}$

GND gives $O(n \log n)$ fill
$O(n^{3/2})$ work.
**Gilbert-Tarjan**

**Graph Contraction**

- edge contraction
- remove mult edges

**Definition**

$G$ is sparse-contractible if every $m$-node contraction of $G$ has $O(m)$ edges.

**Note**

Planar graphs are sparse-contractible.

**Note**

$P_m \times P_m \times P_1$ is not sparse-contractible.

It has a $K_{vn}$ minor.
Example: Let $P_n$ be path graph
$S_n$ be star graph

$G_k = P_k \times S_k$

Claim: All subgraphs of $G_k$ have $\sqrt{n}$-separators
of in GT

Claim: END ordering for $G_k$ with fill $\mathcal{N}(n^{5/4})$

$n = k^2$

Order:
1) Top level sep = "spine" of $G_k$

This gives $k$ paths after removing spine.

2) For each path $P_k$ pick middle $\sqrt{k}$ nodes.

3) For each middle $\sqrt{k}$ nodes pivot "outside in".

4) Pick remaining node any way you like
Consider one wing from the spine

\[ K - \sqrt{K} \]

As we pivot the \( K \) nodes we get \( (K - \sqrt{K})(n^{3/2}) \) fill per wing \( \Omega(n^{3/2}) \) fill.

Over all wings \( \Omega(K^{5/2}) = \Omega(n^{5/4}) \) fill.

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Minimal separator version
Min Degree Heuristic
Fractals

MDH = Pivot on a variable with minimum degree.

Question: How do we break ties?

Consider MDH in planar case.

Simple fill model for planar graphs:

1) Start with a planar graph (genus g surface)
2) Pivot = remove a vertex & its edges
3) Fill at a given time
   \{(x,y) | x & y share a face\}
Fractals

One way to generate them is:
1) Start with infinite grid plane
2) remove a "periodic" subset of vertices

Main Issue: what is area to boundary?

For an $N \times N$ sq $A = N^2$ $B \approx N$

$$\text{dim bdry} = \left( \frac{\log \left( \frac{\text{length}}{\text{area}} \right)}{\log N} \right)^2 = \left( \frac{\log N}{\log N^2} \right)^2 = 1$$

First case: $A_{i+1} = 7A_i$ & $B_{i+1} = 3B_i$

$$\text{dim bdry} = \left( \frac{\log \left( \frac{3N^2}{14N^2} \right)}{\log \left( \frac{3N^2}{14N^2} \right)} \right)^2 = \left( \frac{\log 3}{\log 7} \right)^2 = \frac{\log 3}{\log 7} \approx 1.16$$
Fill in linear case

\[ A_{in+1} = 4A_i \Rightarrow \text{# rounds } \log_4 n \]

\[ B_{in+1} = 2B_i \]

Find base size: \[ 2^{\log_4 n} = n \frac{\log_4^2}{\sqrt{n}} = \sqrt{n} \]

\[ \text{fill } \geq (\sqrt{n})^2 = n \]
Fill for Hex example

\[ A_{i+1} = 7A_i \quad \text{if round} = \log_7 n \]
\[ B_{i+1} = 3B_i \quad \text{boundary size} = 3 \log_7 n \]

\[ \text{fill} = (3 \log_7 n)^2 = (n \log_3 3)^2 = n \log_3 9 \approx n^{1.12} \]

\[ A_{i+1} = 3A_i \]
\[ B_{i+1} = 2B_i \]

\[ \text{fill} = (n \log_3 3)^2 = n \log_3 4 \approx n^{1.26} \]
Diagonal cuts for 2D-Mesh

Vertical

\[ A = (\sqrt{2} K)^2 = 2 K^2 \]
\[ B = 4K \]
\[ \text{fill} = 16 K^2 \]

\[ A = K^2 \]
\[ B = 2K \]
\[ \text{fill} = 4K^2 \]
4-time better