More Direct Methods

Spectral
10/13/09

$G=(V,E)$

Def $S \subseteq V$ is an $f(n)$-separator if (vertex)

1) $|S| \leq f(n)$
2) $\exists$ partition A, B of $V-S$
3) $|A|, |B| \leq \frac{2}{3}n$

Gilbert-Tarjan $S \subseteq V$ is an (α,β)-$f(n)$-separator

$\alpha < 1$ & $\beta > 0$

1) $|S| \leq \beta f(n)$
2) $\exists$ part A, B of $V-S$
3) $|A|$ & $|B| \leq \alpha n$

Lipton-Tarjan If G is planar $\exists$ $\sqrt{n}$-separator

$\alpha = \frac{2}{3}, \beta = \sqrt{8}$

Conjecture $\beta = \sqrt{\pi}$ works.

Pivot Strategies

1) (Nested Disection)(ND)

   1) Find a vertex separator S of G.
   2) Let $G_1, \dots, G_k$ be connected components of $G-S$
   3) Return $ND(G_1), \dots, ND(G_k), S$

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2) (Lipton-Rose-Tarjan)(Generalized ND)

Procedure $LRT(G, H \subseteq V)$

* Returns an ordering of $V-H$
* initially $H = \emptyset$

   1) Find vertex separator S of G & part A, B
   2) Let

      $G_1 = (A \cup S, E(A \cup S) - E(S))$ $H_1 = S \cup (H \cap A)$

      $G_2 = (B \cup S, E(B \cup S) - E(S))$ $H_2 = S \cup (H \cap B)$

   3) return

      $LRT(G_1, H_1), LRT(G_2, H_2), S-H$

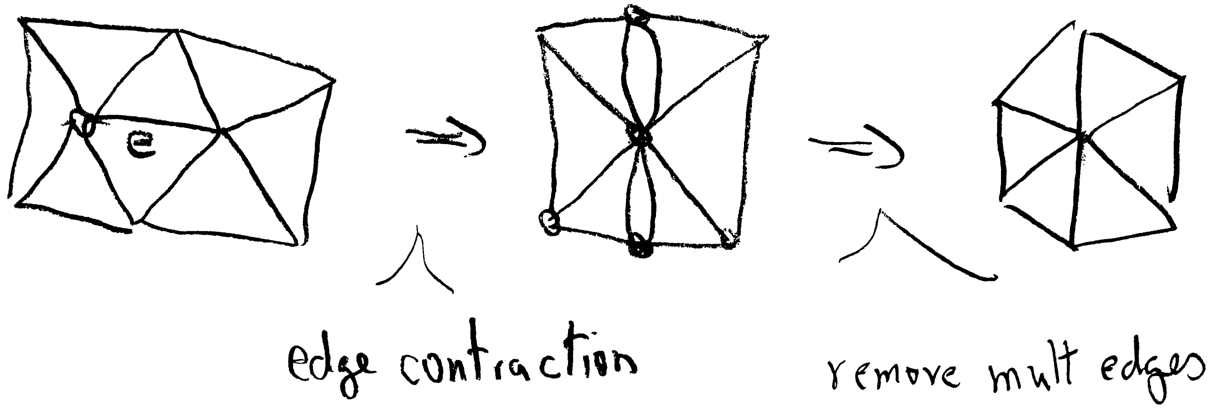
Fill and Work Results

$\mathcal{S}_\alpha$ be a class of graph closed under subgraphs
s.t.
1) Each graph has an n^α-separator.

<u>Thm</u> (LRT) For $G \in \mathcal{S}_{1/2}$
GND gives $O(n \log n)$ fill
$O(n^{3/2})$ work.

Gilbert-Tarjan

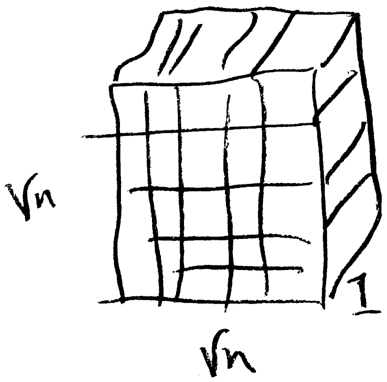
Graph Contraction



<u>Def</u> G is sparse-contractible if every m-node contraction of G has $O(m)$ edges.

<u>Note</u> Planar graphs are sparse-contractible

<u>Note</u> $P_{\sqrt{n}} \times P_{\sqrt{n}} \times P_1$ is not sparse-contractible



It has a $K_{\sqrt{n}}$ minor.

<u>Example</u> Let P_n be path graph
S_n be star graph

$G_K = P_K \otimes S_K$

<u>Claim</u> All subgraphs of G_K have $\sqrt{n}$-separators

pf in GT

<u>Claim</u> $\exists$ ND ordering for G_K with fill $\Omega(n^{5/4})$

$n = K^2$

<u>order</u>

1.) Top level sep $\equiv$ "spine" of G_K

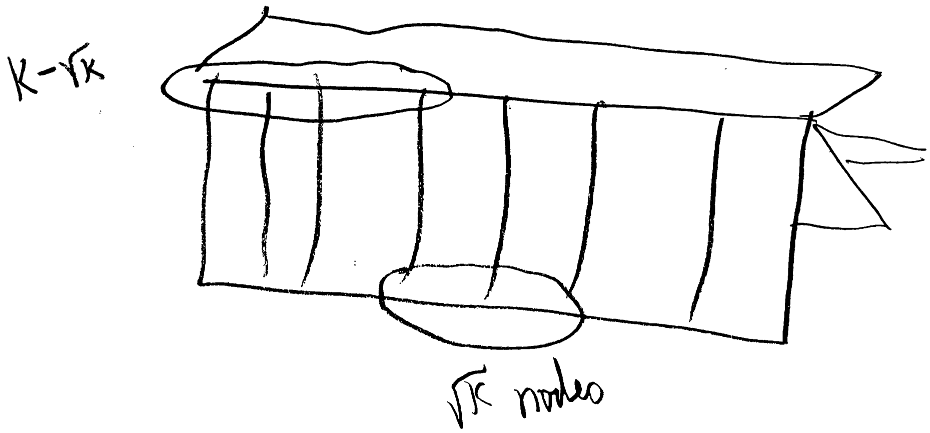
This gives k paths after removing spine.

2) For each path P_K pick middle $\sqrt{K}$ nodes.

3) For each middle $\sqrt{K}$ nodes pivot "outside in".

4) Pick remaining node any way you like

Consider one wing from the spine

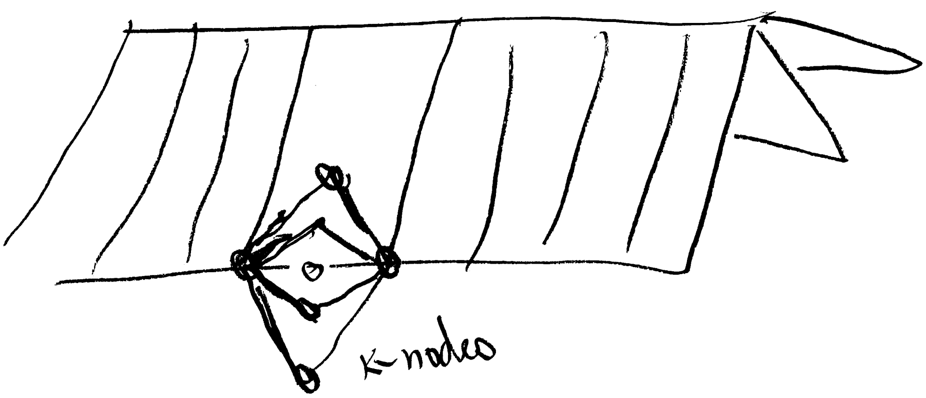


As we pivot the $\sqrt{K}$ nodes we get $(K-\sqrt{K})(\sqrt{K}/2)$ fill

per wing $\Omega(K^{3/2})$ fill

Over all wings $\Omega(K^{5/2}) = \Omega(n^{5/4})$ fill.

Minimal separator version



Min Degree Heuristic
Fractals

MDH $\equiv$ Pivot on a variable with minimum degree.

Question How do we break ties?

Consider MDH in planar case.

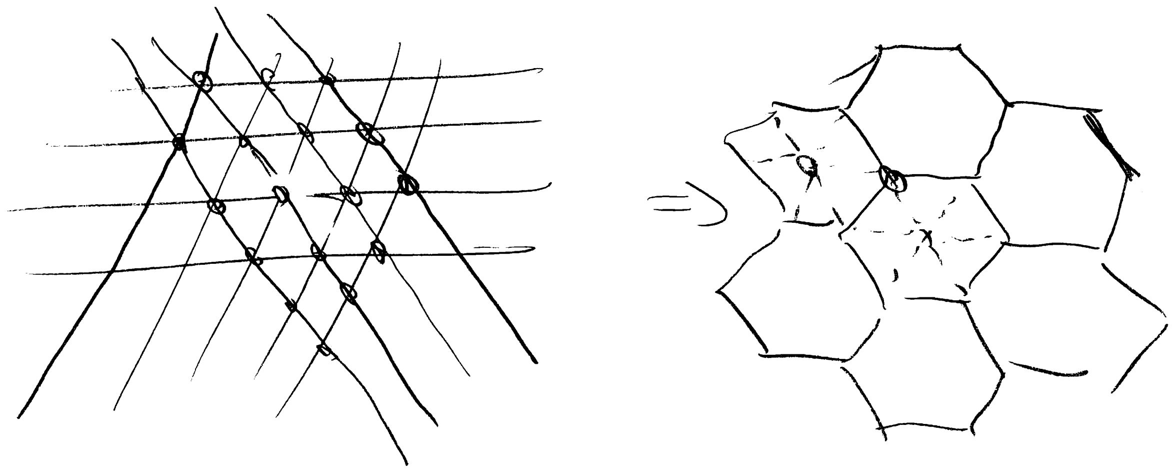
Simple fill model for planar graphs

1) Start with Δ planar graph (genus g surface)
2) Pivot $\equiv$ remove a vertex & its edges
3) Fill at a given time
   $\{(x,y) \mid x \,\&\, y \text{ share a face}\}$

Fractals

One way to generate them.

1) Start with infinite Δed plane
2) remove a "periodic" subset of vertices



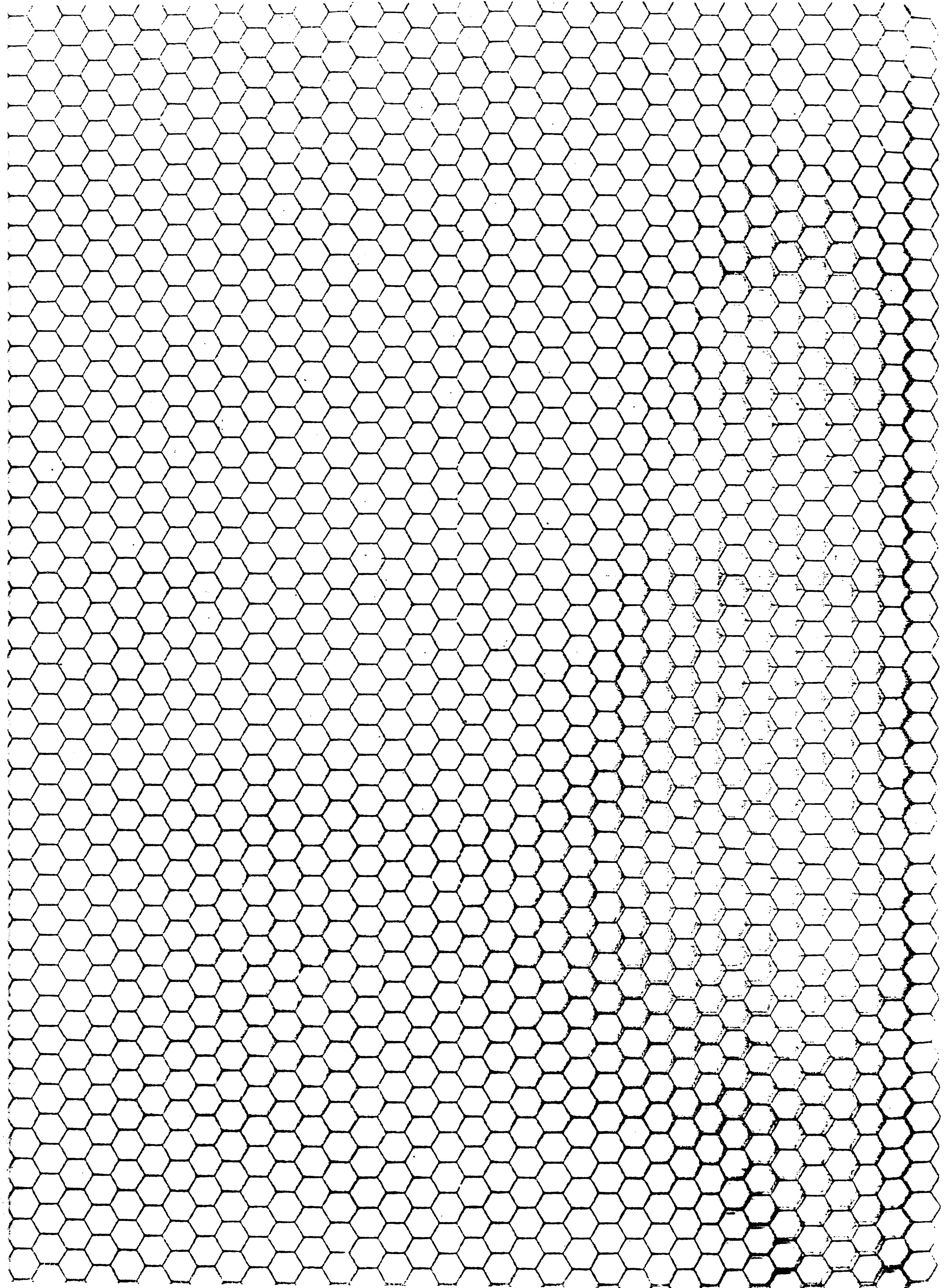
Main Issue what is area to bdry?

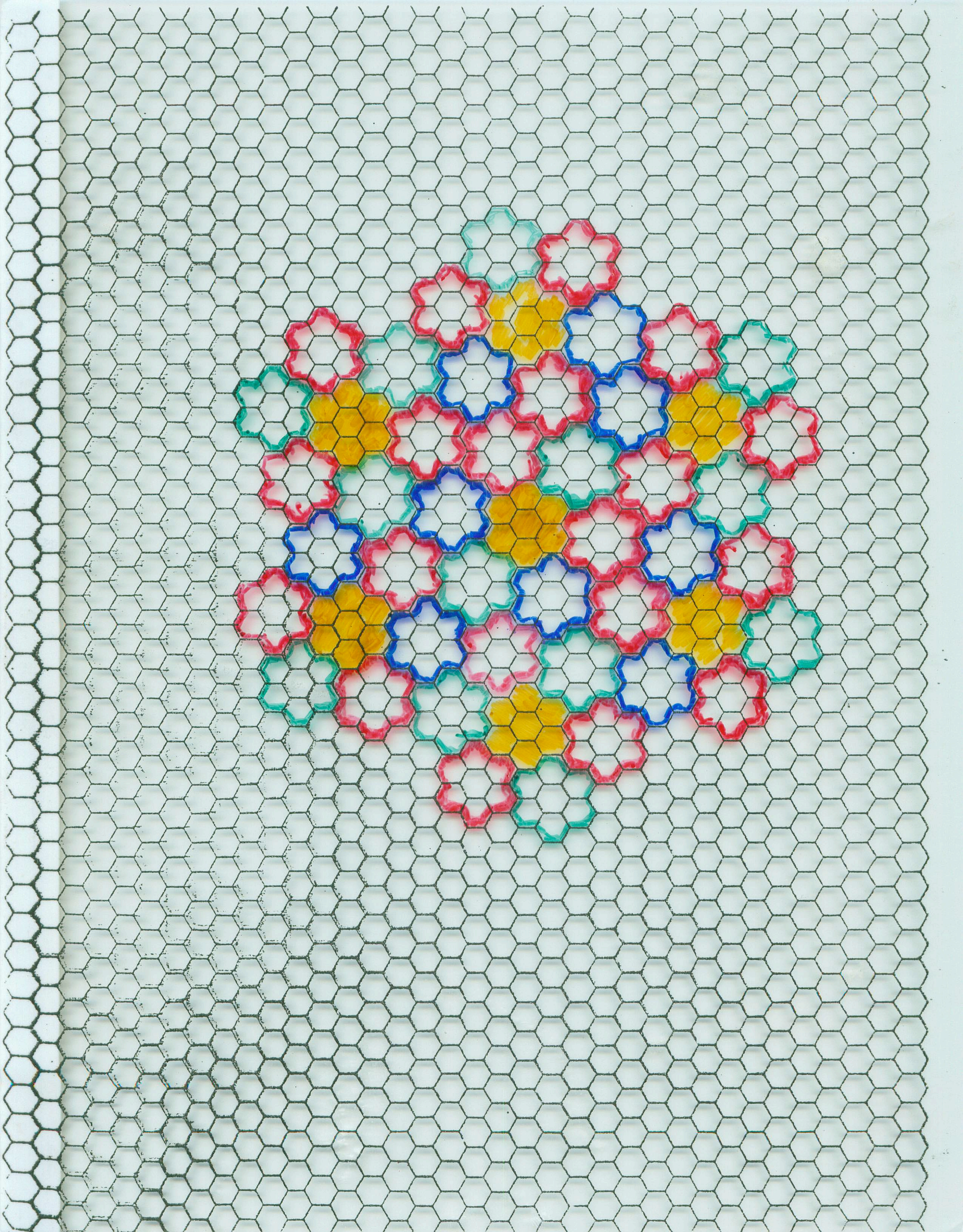
For an $N \times N$ sq $A = N^2$ $B \approx N$

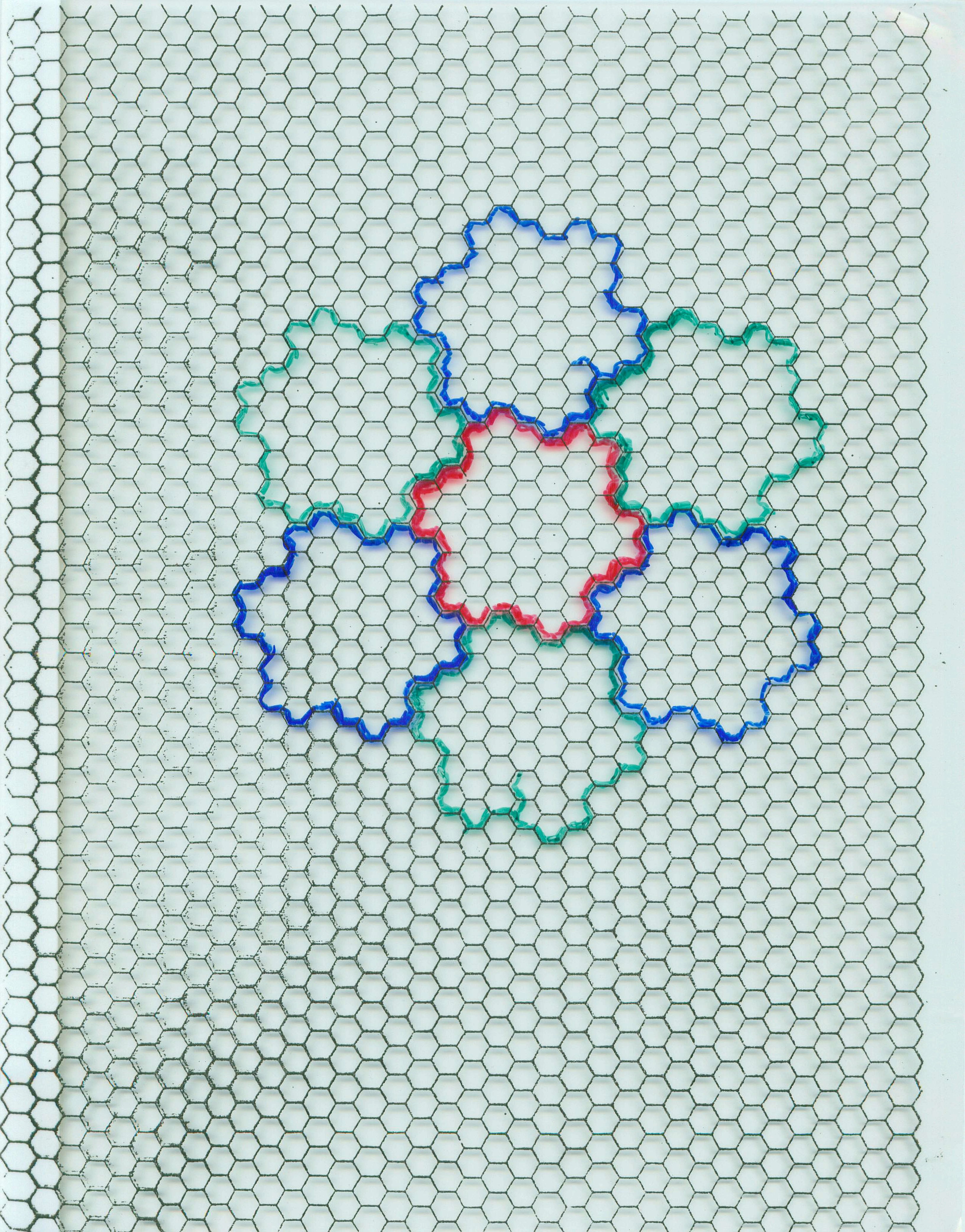
$$\text{dim bdry} = \left(\frac{\log(\text{length})}{\log(\text{area})}\right)^2 = \left(\frac{\log N}{\log N^2}\right)^2 = 1$$

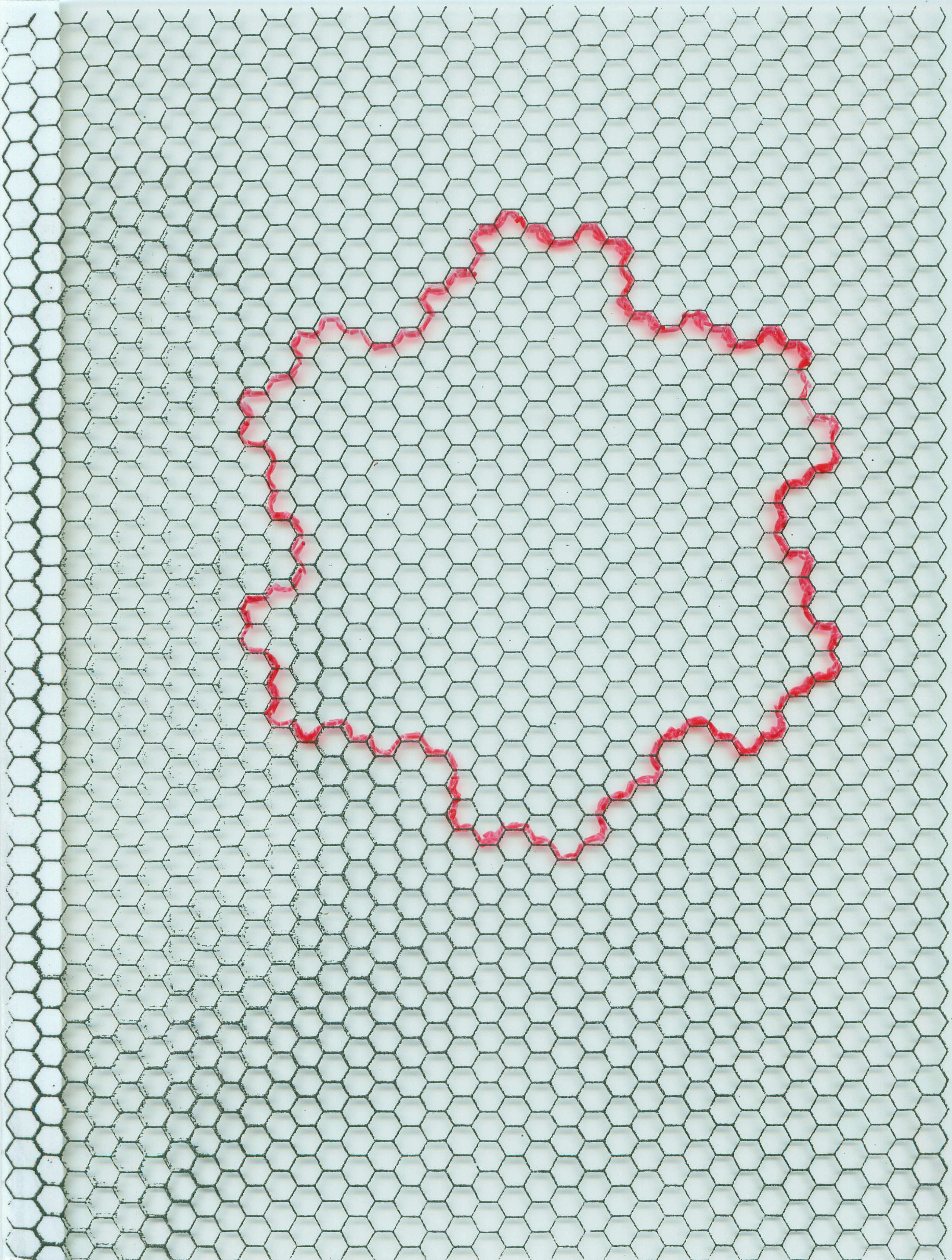
First Hex case $A_{i+1} = 7A_i$ & $B_{i+1} = 3B_i$

$$\text{dim bdry} = \left(\frac{\log(3^K)}{\log(7^K)}\right)^2 = \left(\frac{K\log 3}{K\log 7}\right)^2 = \frac{\log 9}{\log 7} = 1.16$$

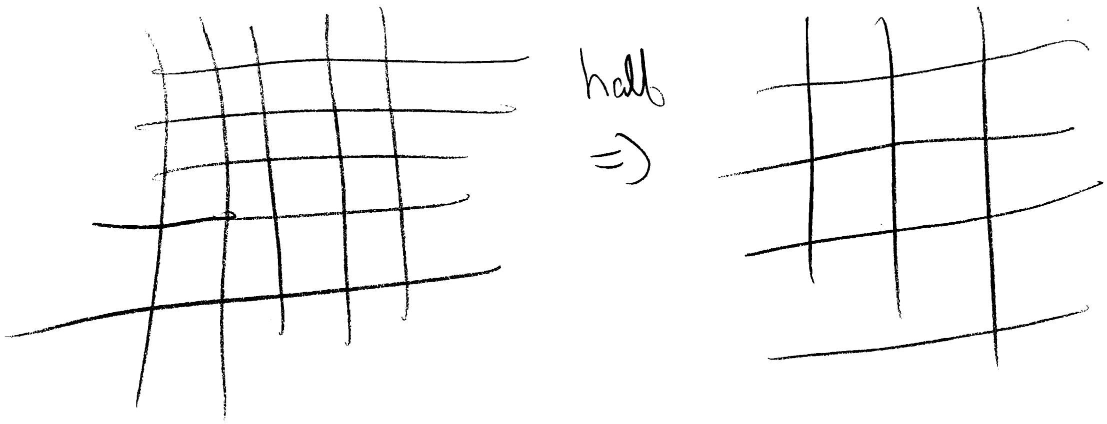








Fill in linear case



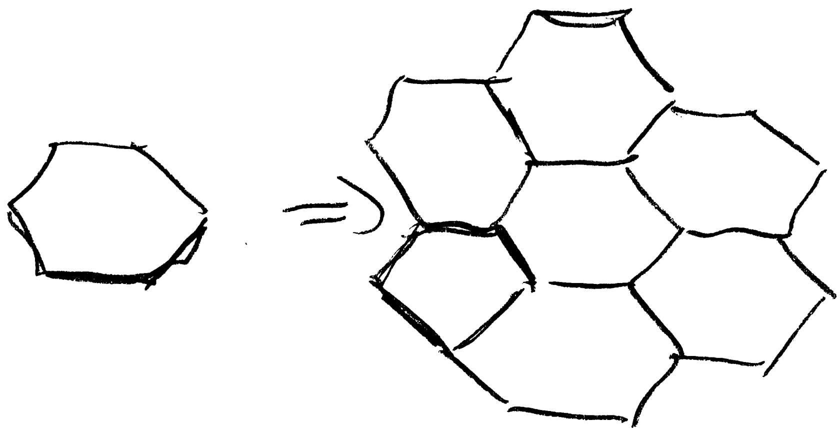
$$A_{i+1} = 4A_i \Rightarrow \#\text{rounds } \log_4 n$$

$$B_{i+1} = 2B_i$$

final bdry size $\quad 2^{\log_4 n} = n^{\log_4 2} = \sqrt{n}$

fill $\geq (\sqrt{n})^2 = n$

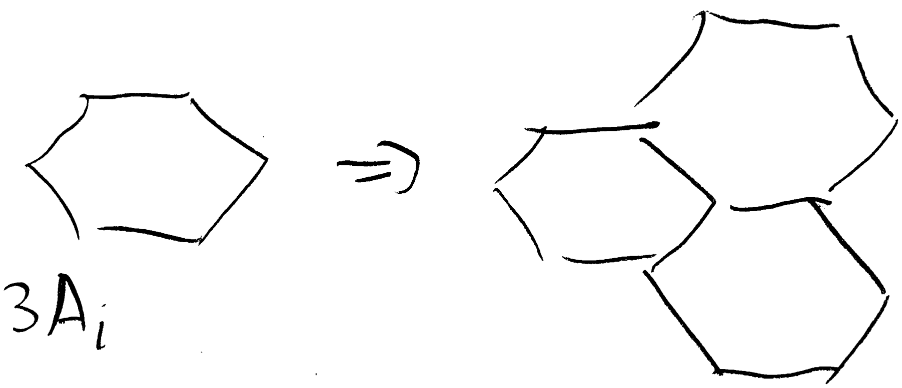
Fill for Hex example



$A_{i+1} = 7A_i$ # rounds $= \log_7 n$

$B_{i+1} = 3B_i$ bdary size $= 3^{\log_7 n}$

$$\text{fill} = \left(3^{\log_7 n}\right)^2 = \left(n^{\log_7 3}\right)^2 = n^{\log_7 9} \approx n^{1.12}$$



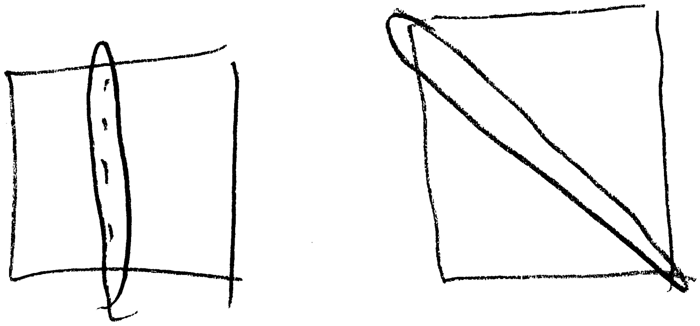
$A_{i+1} = 3A_i$

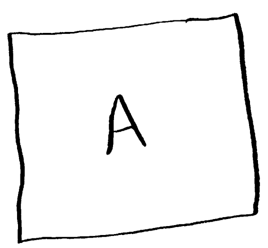
$B_{i+1} = 2B_i$

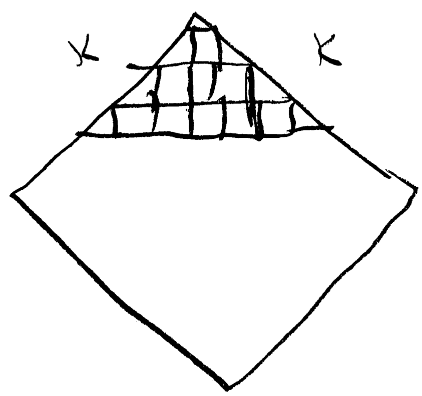
$$\text{fill} = \left(n^{\log_3 2}\right)^2 = n^{\log_3 4} \approx n^{1.26}$$

Diagonal cuts for 2D-Mesh

Vertical







$B = 4K$

$\text{fill} = 16K^2$

$A = (\sqrt{2}K)^2 = 2K^2$

$B = 4K$

normalize

$A = K^2$

$B = 2K$

$\text{fill} = 4K^2$

4-time better