Spectral
10/1/09

Estimating λ_2 for Laplacian's

$G = (V, E, w) \quad L = D - A = L(G)$

eg

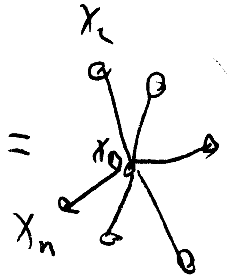
$G = K_n \quad L = \begin{pmatrix} n-1 & & -1 \\ & \ddots & \\ -1 & & \ddots \\ & & & n-1 \end{pmatrix}$

$$L \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{u_1} = \begin{pmatrix} n \\ -n \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad L u_1 = n u_1 \qquad u_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Big\} i$$

$j = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ eigenvector $j, u_1, \ldots, u_{n-1}$

value $0, n, \ldots, n$

We have full set

$G = S_n =$ 

$|V| = n+1$

$$u_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} i \\ \\ \\ \\ \\ \end{matrix} \qquad i = 1, \dots, n$$

$$u = \begin{pmatrix} \alpha \\ \beta \\ \vdots \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} n & -n \\ -n & n \end{pmatrix} x = \lambda \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} x$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} x = \lambda/n \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} x$$

$G = P_n$ Path graph

Claim $\lambda_2 = \min\limits_{\substack{v \perp \mathbb{1} \\ v \neq 0}} \frac{v^T L v}{v^T v}$

$(\geq)$ $Lx = \lambda_2 x \Rightarrow x^T L x = \lambda_2 x^T x$ when $x \perp \mathbb{1}$ & $x \neq 0$

$(\leq)$ Spectral Thm

$$x = \left(-\tfrac{n}{2}, -\tfrac{n}{2}+1, \ldots, +n/2\right)$$

$$x^T L x = \Sigma (x_i - x_{i+1})^2 = n-1$$

$$x^T x = 2 \sum_{i=1}^{n/2} i^2 \approx 2(n/2)^3/3 = \frac{n^3}{12}$$

$$\lambda_2 \leq \frac{n-1}{n^3/12} \leq \frac{12}{n^2}$$

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Estimating λ_2

Upper Bounds

Thm A is real sym with eigenvalues $\lambda_1 \cdots \lambda_n$

eigenvectors $x_1 \cdots x_n$

$$\lambda_k = \min_{x \perp x_1 \cdots x_{k-1}} \frac{x^T A x}{x^T x}$$

Cor A is a graph Laplacian $x_1 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = j$

$$\lambda_2 = \min_{x \perp j} \frac{x^T A x}{x^T x}$$

Last time λ_2 for P_n (path graph) $\leq 12/n^2$

Consider a partial order on real sym matrices

Def $A \succcurlyeq 0$ if A is positive semidefinite

iff $\lambda_1 \cdots \lambda_n \geq 0$

iff $x^T A x \geq 0$

Def $A \succeq B$ if $x^TAx \geq x^TBx$

iff $x^T(A-B)x \geq 0$

Note $A \succeq B$ & $B \succeq C \Rightarrow A \succeq C$

Thm $A \succeq B$ then $\lambda_k(A) \geq \lambda_k(B)$

pf

Use Courant-Fischer: $\lambda_k(A) = \min_{\dim(S)=k} \max_{x \in S} \frac{x^TAx}{x^Tx}$

$$\lambda_k(A) = \min_{\dim(S)=k} \max_{x \in S} \frac{x^TAx}{x^Tx} \geq \min_{\dim(S)=k} \max_{x \in S} \frac{x^TBx}{x^Tx}$$

$$= \lambda_k(B)$$

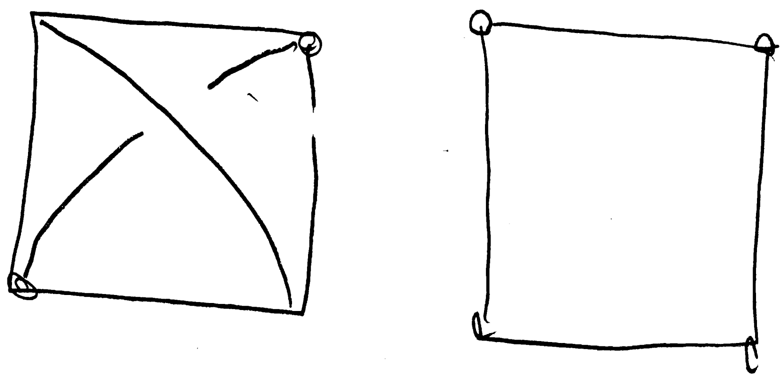
Note $c \in \mathbb{R}$ & A real-sym then $\lambda_k(cA) = c\lambda_k(A)$

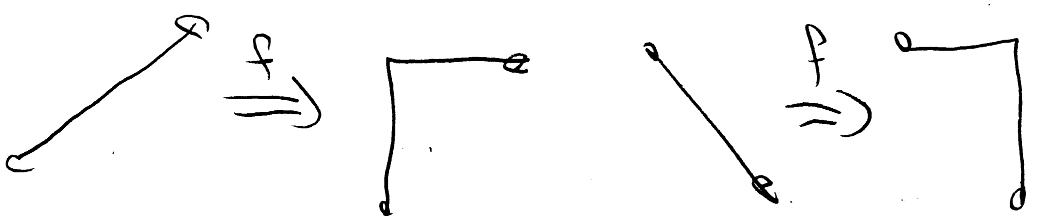
Def Support of A by B, $\sigma(A/B) = \min_c cB \succeq A$

Def G (guest), H (host)

Path Embedding: $f : E_G \to$ Paths in H

eg $G = K_4 \quad H \equiv C_4$



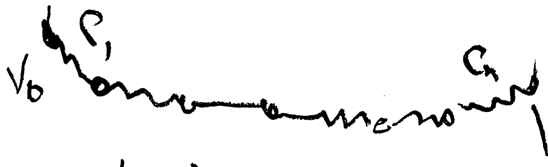


Conjestion $\equiv \max\limits_{e \in H} |P| \; e \in \text{Path} \}$

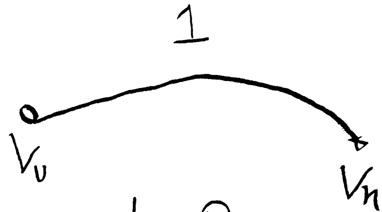
Dilation $\equiv \max\limits_{P} \{|P|\}$

In example: Conj = 3 & Dil = 2

Case of a path and an edge



$L \equiv A$



$L \equiv B$

conductors c_1 — — c_n

resistors $R_1 = 1/c_1$ — — $R_n = 1/c_n$

$$R = \sum_{i=1}^{n} R_i$$

<u>Claim</u> Effective resistance is R

ie $x_0, \ldots, x_n$ is a voltage setting

<u>Claim</u> $R \cdot x^T A x \geq x^T B x$ ie $R \cdot A \succeq B$

ie $D_i = X_i - X_{i-1}$ (voltage drop)

$$\left(\sum_{i=1}^{n} D_i\right)^2 \leq R \cdot \sum c_i D_i^2 \qquad (*)$$

Recall Cauchy-Stwartz $a, b \in \mathbb{R}^n$

$$(a^T b)^2 = (\cos^2 \theta)(a^T a)(b^T b)$$

ie $(a^T b)^2 \leq (a^T a)(b^T b)$

Set $a = (\sqrt{R_1}, \dots, \sqrt{R_n})$ $b = (\sqrt{C_1}\, D_1, \dots, \sqrt{C_n}\, D_n)$

$$\left(\sum D_i\right)^2 = (a^T b)^2 \leq (a^T a)(b^T b) = R \sum C_i D_i^2$$

Claim $\exists V$ st $(*)$ equality

Set $a = b$ by setting $\sqrt{R_i} = \sqrt{C_i}\, D_i$

ie $R_i = D_i$

Mediant of Fractions

Fractions $\left(\frac{a_1}{b_1}, \dots, \frac{a_n}{b_n}\right)$

$$\text{mediant} \equiv \frac{\sum a_i}{\sum b_i}$$

Claim: $\dfrac{\sum a_i}{\sum b_i} \le \max_i \dfrac{a_i}{b_i}$ $\quad a_i > 0 \quad b_i > 0$

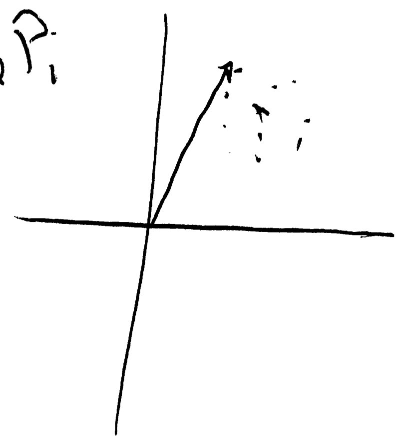
picture proof

Consider pionts in $\mathbb{R}^2$ $\underset{P_1}{(b_1, a_1)}, \dots, \underset{P_n}{(b_n, a_n)}$

Let $\bar{P}$ average or center of mass.

$\text{Slope}(P_i) = \dfrac{a_i}{b_i}$ $\quad \text{Slope}(\bar{P}) = \dfrac{\sum a_i}{\sum b_i}$

$\text{Slope}(\bar{P}) \le \max \text{Slope } P_i$



Laplacian's as sums of Laplacian's

$$E_{ij} = \begin{array}{cc} & \begin{array}{ccccc} & i & & j & \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \begin{pmatrix} 0 & & 0 & & 0 \\ & 1 & & -1 & \\ 0 & & 0 & & 0 \\ & -1 & & 1 & \\ 0 & & 0 & & 0 \end{pmatrix} \end{array}$$

$i < j$

Def A is diagonal dominate if $A_{ii} \geq \sum_{j \neq i} |A_{ij}|$

Claim $L = L(G)$ then $L = \sum_{(i,j) \in E} w_{ij} E_{ij}$

Claim A is SDD then $A = \sum_{\substack{i \neq j \\ A_{ij} < 0}} -A_{ij} E_{ij} + \sum_{\substack{A_{ij} > 0 \\ i \neq j}} A_{ij} P_{ij} + D'$

$$P_{ij} = \begin{array}{cc} & \begin{array}{cc} i & j \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{array}$$

& D' is non-neg diagonal.

<u>Thm</u> $f: G \to H$ with conjestion c & dilation d

then $c d L_H \succeq L_G$

<u>pf</u> $L(G) = G \quad L(H) = H$

$E_1 \cdots E_m$ edge subgraphs of G

$P_1 \cdots P_m$ the path of $H \qquad \sum P_i \preceq c \cdot H$

$$\frac{x^T G x}{x^T H x} \le \frac{x^T \sum E_i x}{(1/c) x^T \sum P_i x} = \frac{c \sum x^T E_i x}{\sum x^T P_i x} \le c \cdot \mathrm{Max} \frac{x^T E_i x}{x^T P_i x}$$

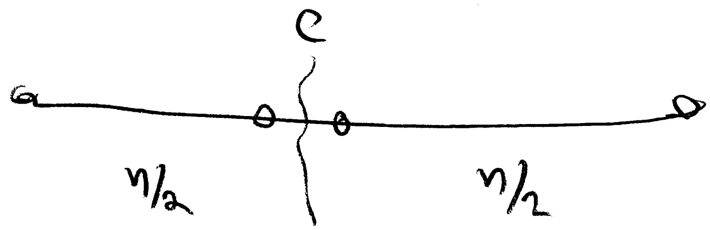
$$\le c \cdot d$$

ie $\quad c \cdot d\, x^T H x \ge x^T G x$

Back to λ_2 for P_n

$f: K_n \xrightarrow{\text{path}} P_n$

Congestion



$$C = \left(\frac{n}{2}\right)^2$$

dilation $\equiv n-1$

By Thm: $\lambda_2(K_n) \leq c \cdot d \cdot \lambda_2(P_n)$

$$\frac{n}{c \cdot d} \leq \lambda_2(P_n)$$

$$\frac{n}{\left(\frac{n}{2}\right)^2(n-1)} = \frac{4}{n(n-1)} \leq \lambda_2(P_n)$$

Estimating λ_2 for C_n^k using path embeddings

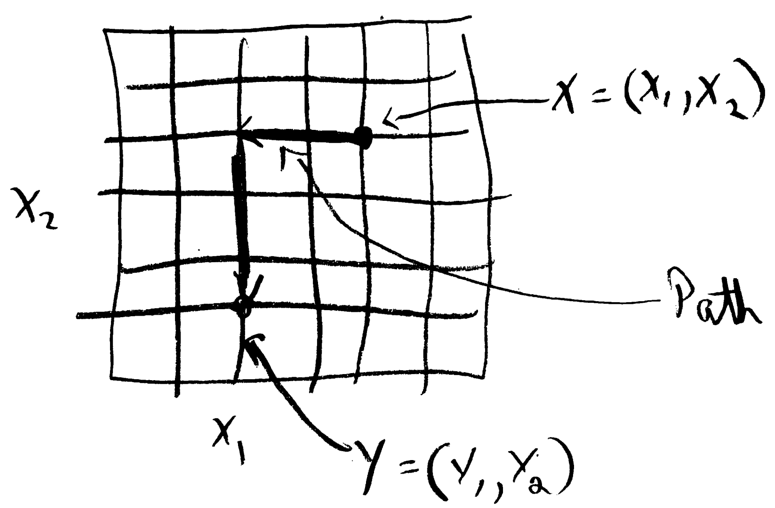
<u>Recall</u> $C_n^k \equiv \underbrace{C_n \otimes \cdots \otimes C_n}_{k}$ Cartisian Product.

Vertices are $X = (X_1, \ldots, X_k)$ $X_i \in \{0, \ldots, n-1\}$

We first consider $P_n^k = P_n \otimes \cdots \otimes P_n$

<u>Def</u> The path $P(X,Y) \equiv$ correct components from left to right.

Let work out $k=2$

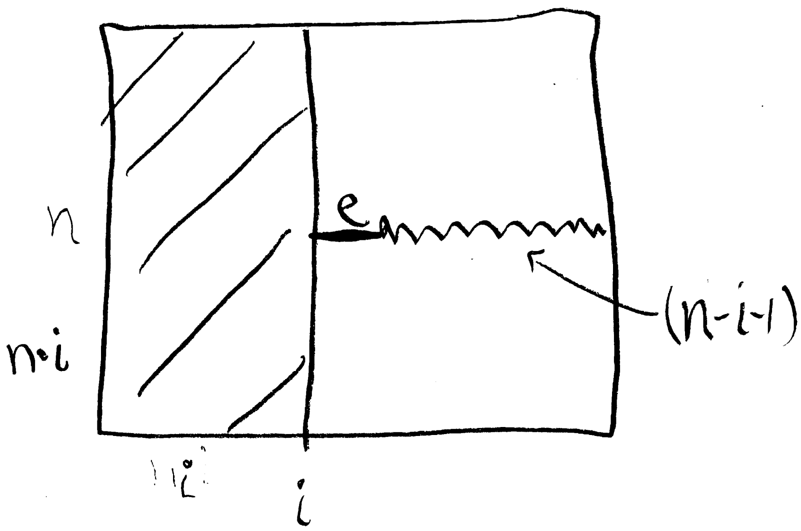


dilation $= 2(n-1)$

conjestion:

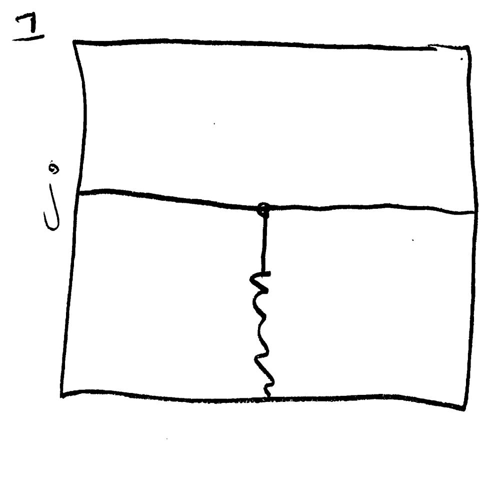
Case of horizontal edge

$\mathrm{Conj}(e) \leq (n \cdot i)(n-i)$



vertical edge e'

$\mathrm{Conj}(e') \leq n \cdot j\,(n-j)$



therefore $C \leq n^3$

$C \cdot d \approx n^4$

$\lambda_2 \geq \frac{n^2}{n^4} = \frac{1}{n^2}$

General Case

An edge $\left\{ \begin{array}{l} (x_1, \ldots x_i \ldots, x_k) \\ (x_1, \ldots, x_i + 1, \ldots x_k) \end{array} \right\} = e$

Conj Start vertices $(*, *, \ldots *, y_i, x_{i+1} \cdots x_k)$

$y_i \leq x_i$

End vertices $(x_1, \ldots x_{i-1}, y_i, *, \ldots, *)$

$x_i < y_i$

$\text{Conj} \leq n^{k+1}$

$\text{Dilation} \leq kn$

$\Rightarrow \lambda_2 \geq \frac{n^k}{k \cdot n \cdot n^{k+1}} = \frac{1}{kn^2}$

Thus

$$\Omega\left(\frac{1}{kn^2}\right) = \lambda_2(C_n^k) = O\left(\frac{1}{n^2}\right)$$

The right answer is $\lambda_2 = \Theta\left(\frac{1}{n^2}\right)$

$\lambda_2(C_n^k)$ & mixing rate of C_n^k

Note C_n^k is regular of degree $2k$.

Let A adj matrix of C_n^k

Transition matrix is $A/2k = M$

$L = (2kI - A)$ $\lambda_2(L) = \lambda$ $L/2k = I - A/2k = I - M$

$\lambda_2(\frac{1}{2k}L) = \lambda/2k$

$\lambda_2(M) = 1 - \lambda/2k$ (second largest)

$\therefore$ Mixing rate $1 - \lambda/2k \approx \left(1 - \frac{1}{kn^2}\right)$