Spectral
9/29/09

Springs and Graph Laplacian's

Input: Graph of Springs (mattress)

$G = (V, E, k)$

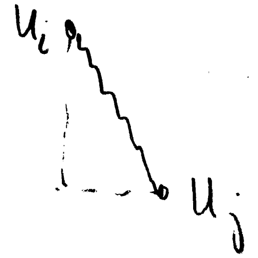
$k_{ij} \equiv$ spring constant for edge E_{ij}

Consider only vertical displacements

$u_i \equiv$ displacement of V_i

Goal: Find solutions to Newton: $f = ma$

Find forces for displacement $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$.



Set force on u_i by spring E_{ij}

$-(u_i - u_j) k_{ij}$

ie linear spring model.

$L = L(G)$

Force vector $-Lu$

Let m_i = mass of V_i $\quad M = \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_n \end{pmatrix}$

$m_i > 0$

View u as a fcn of time $u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}$

Acceleration: For $u_i(t) = \dfrac{d^2 u_i}{dt^2}$

$$\frac{du}{dt} = \begin{pmatrix} \frac{du_1}{dt_1} \\ \vdots \\ \frac{du_n}{dt_n} \end{pmatrix} \quad \therefore \quad a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \frac{d^2 u}{dt^2}$$

Newton: $-Lu = M\left(\dfrac{d^2 u}{dt^2}\right)$

two ways to solve:

1) Solve for u given initial conditions say $u(0)$ & $u'(0)$

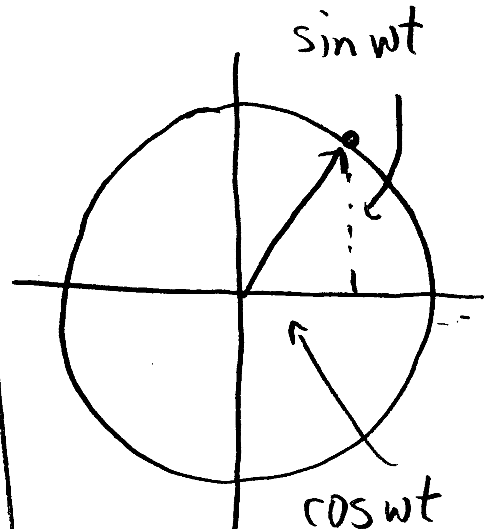
2) Find steady state solutions

Do 2) using Guess and check.

Recall: $e^{iwt} \equiv \cos wt + i \sin wt$

$$\frac{de^{iwt}}{dt} = iw\, e^{iwt} \overset{\text{or}}{\equiv} -w \sin wt + iw \cos wt$$

$$= iw(\cos wt + i \sin wt)$$



$$\frac{d^2 e^{iwt}}{dt^2} = \frac{d(iw\, e^{iwt})}{dt} = i^2 w^2 e^{iwt}$$

$$= -w^2 e^{iwt}$$

$$\frac{d \cos x}{dx} = -\sin x$$

$$\frac{d \sin x}{dx} = \cos x$$

Guess: $u = e^{i\omega t}x$ some vector x.

$$\frac{d^2u}{dt} = -\omega^2 e^{i\omega t} x$$

Check:

$$M(-\omega^2 e^{i\omega t} x) \stackrel{?}{=} -L(e^{i\omega t} x)$$

iff $\omega^2 M x = Lx$ set $\lambda = \omega^2$

iff $Lx = \lambda M x$ ie (λ, x) is a generalized eigen-pair.

Claim: Eigenvalues of $Lx = \lambda M x$ are real nonneg.

Change of variables $y = M^{1/2} x$ or $x = M^{-1/2} y$

$$L M^{-1/2} y = \lambda M M^{-1/2} y$$

$$\underbrace{M^{-1/2} L M^{-1/2}}_{L'} y = \lambda \cancel{M^{-1/2} M M^{-1/2}} y$$

L' is positive semidefinite

$\therefore$ we have found a space of dim n solutions.

eg $w_i = \sqrt{\lambda_i}$ then X_i the eigenvector

$$u(t) = \alpha_1 e^{i w_1 t} X_1 + \cdots + \alpha_n e^{i w_n t} X_n$$

is a solution.

Consider setting mass = the weighted degree

Eigenpairs $\mathcal{L}X = \lambda D X$ $(*)$ $\mathcal{L} = D - A$

change of variable $Y = DX$

$(*)$ iff $(D-A)D^{-1}Y = \lambda Y$

$(I - AD^{-1})Y = \lambda Y$ $M = AD^{-1}$ transition matrix for a random walk.

iff $Y - MY = \lambda Y$

iff $-MY = (\lambda - 1)Y$ iff $MY = (1-\lambda)Y$

If $0 = \lambda_1 < \lambda_2 \cdots \leq \lambda_n$ eigenvalues of $(*)$

then $0 \leq 1-\lambda_n, \ldots, 1-\lambda_{n-1}, 0$ eigen of $MX = \lambda X$

Estimating λ_2 for Laplacian's

$$G = (V, E, w) \qquad L = D - A = L(G)$$

eg

$$G = K_n \qquad L = \begin{pmatrix} n-1 & & -1 \\ & \ddots & \\ -1 & & n-1 \end{pmatrix}$$

$$L \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} n \\ -n \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad L u_1 = n u_1 \qquad u_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Big\} i$$

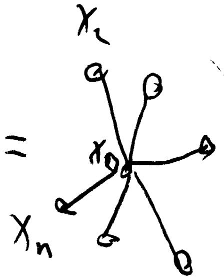
(where $u_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$)

$$j = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

eigenvector $j, u_1, \ldots, u_{n-1}$

value $0, n, \ldots, n$

We have full set

$G = S_n =$  $|V| = n+1$

$$u_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad i = 1, \dots, n$$

$$u = \begin{pmatrix} \alpha \\ \beta \\ \vdots \\ \vdots \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} n & -n \\ -n & n \end{pmatrix} x = \lambda \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} x$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} x = \lambda/n \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} x$$

$G = L_n$ Line graph

Claim $\lambda_2 = \min\limits_{\substack{V \perp 1 \\ V \neq 0}} \frac{V^T L V}{V^T V}$

$(\geq)$ $LX = \lambda_2 X \Rightarrow X^T L X = \lambda_2 X^T X$ when $X \perp 1$ & $X \neq 0$

$(\leq)$ Spectral Thm

$X = (-\frac{n}{2}, -\frac{n}{2}+1, \ldots, +n/2)$

$x^T L x = \Sigma(x_i - x_{i+1})^2 = n-1$

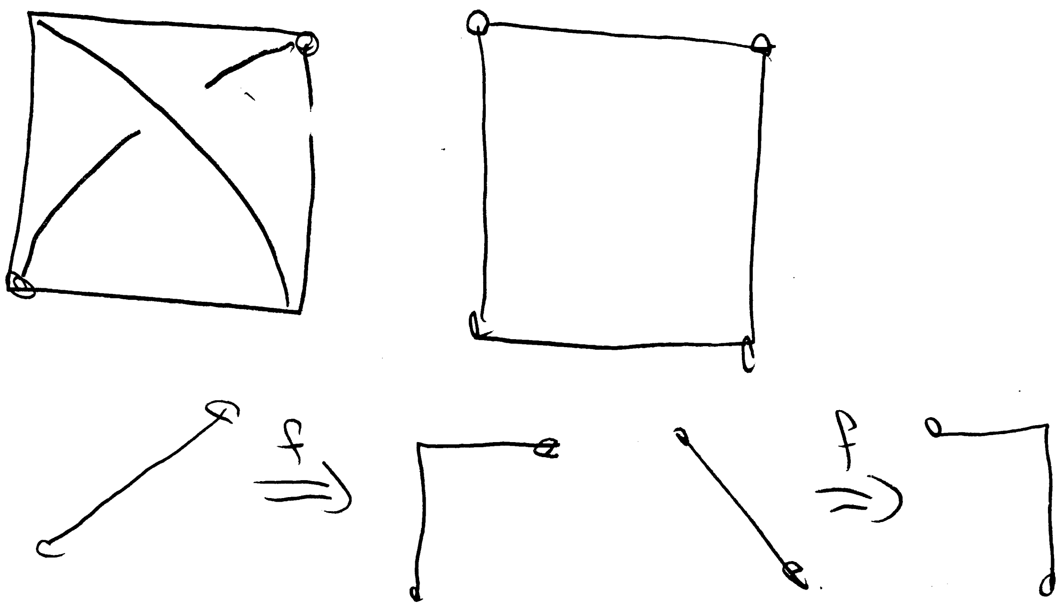
$x^T x = 2 \sum_{i=1}^{n/2} i^2 \approx 2(n/2)^3/3 = \frac{n^3}{12}$

$\lambda_2 \lessapprox \frac{n-1}{n^3/12} \leq \frac{12}{n^2}$

Def G (guest), H (host)

Path Embedding: $f: E_G \to$ Paths in H

eg $G = K_n \quad H = C_4$



Conjestion $\equiv \max\limits_{e \in H} |P| \; e \in \text{Path} \; \}$

Dilation $\equiv \max\limits_{P} \{ |P| \}$

Thm $f: K_n \xrightarrow{\text{path}} G$ with conjestion c & dilation d

then $\frac{n}{c \cdot d} \leq \lambda_2(G)$

Pf $A = L(G)$ $B = L(K_n)$ P_{ij} = Path from i to j

$L_{ij} = L(P_{ij})$

$Ax = \lambda_2 x$

$A \leq \frac{\sum L_{ij}}{c \cdot d}$

$$\frac{\lambda_2}{n} = \frac{x^T A x}{x^T B x} =$$

$$\leq \frac{(x^T \sum L_{ij} x)/c \cdot d}{x^T \sum E_{ij} x} = \frac{1}{cd} \frac{\sum x^T L_{ij} x}{\sum x^T E_{ij} x}$$

$$\leq \operatorname{Max} \frac{x^T L_{ij} x}{x^T E_{ij} x}$$