Random Walks & Mixing Rate

$\Pi(v) = \frac{d(v)}{2m}$ = stationary dist

Mixing Rate

$M = \lim_{t \to \infty} \sup_{i,j} \max_{i \neq j} | \frac{P_{i,j}^{(t)}}{\Pi(i)} - \Pi(i) |^{1/t}$

Method: Coupling

Def: Cartesian Product: $G = (V, E) \& H = (V', E')$

$G \otimes H = (\bar{V}, \bar{E})$

$\bar{V} = V \times V'$

$\langle (x, x'), (y, y') \rangle \in \bar{E}$ if

$(x = x' \& (x, y') \in E') \lor ((x' = y' \& (x, y) \in E))$
\[ L_n = \text{nodes} \]

\[ L_n \times L_m = \]

\[ C_n^k = C_n \times \cdots \times C_n \text{ where } C_n \equiv \text{cycle of length } n. \]

\[ \equiv k \text{-dim Torus (n is odd)} \]

\[ V = \{(x_1, \ldots, x_k) \mid x_i \in \{0, \ldots, n-1\}\} \]

**Step:** 1) Pick a coordinate \(1 \leq i \leq k\)

2) Pick \( \varepsilon \in \{\pm 1\} \)

3) Add \( \varepsilon \) to \(i\)th coord \((\text{mod } n)\)
Two random walks:

(V₁, V₂, ..., Vₙ) starting at some fixed V₁
(W₁, ...) at a random W₁

1) V is a simple random walk, generating 1 ≤ i ≤ k & z ∈ [z₁, z₂]
2) W is "coupled" to V.

If jth coord of Vᵢ = Wᵢ; then W's move to (i, ±z)

The distance from v to w in the jth coord.
Is a random walk moving 2 steps left/right.

Claim: Average hitting time (n²-1)/6

e.g. voltage arguments N=7

Inject 2 units per node but one. Measure average
voltage.

\[ \text{Volts} = (6, 10, 12) \]

\[ \text{Average} = \frac{2}{7} (6 + 10 + 12) \]

\[ \text{Average} = \frac{2}{7} \cdot 28 = 8 \]

\[ = \frac{2² - 1}{6} \]
For fixed \( j \) the expected number of steps till coupled is
\[
\left( \frac{n^3}{6} \right)
\]

Lovasz: Expected number of steps till all coord couple is
\[
k \left( \frac{n^3}{6} \right)
\]

I see: \((k \log k) \left( \frac{n^2}{6} \right)\)

For each \( k \) by \( k \) step we expect to do a step on each coord.

\[\text{Markov } X \geq 0 \text{ random variable} \]
\[
\text{Prob} (X \geq a \text{E}(X)) \leq \frac{1}{a}
\]

\[
\text{Prob} (V_t \neq W_t) \leq \frac{1}{6} \text{ for } t \geq k \log k n^2
\]
\[ t \geq C k \log n^2 \quad \text{then} \quad \text{Prob}(V_t \neq W_t) \leq \left( \frac{1}{6} \right)^c \]

\[
\left| P(V_T \neq S) - \frac{1}{n^k} \right| = \left| P(V_T \neq S) - P(W_T \neq S) \right|
\]
\[
= P(W_T \neq V_T) \leq 6^{-T/k \log n^2}
\]

Minimal Rate \[
= 6^{-\frac{1}{k \log n^2}} < 1 - \frac{1}{k \log n^2}
\]

Note \[ V_n \Rightarrow (1 - \frac{1}{n})^n \geq \frac{1}{e} \]

\[
\lim_{n \to \infty} (1 - \frac{1}{n})^n = \frac{1}{e}
\]
Another Mixing Rate

\( q_t = (q_1, \ldots, q_n) \) Stationary Dist (\( i \sim -1 \))

\[ P^{(+)} = \text{dist at time } t \]

**Mixing Rate**

\[ M = \sup_{p^{(0)}} \lim_{t \to \infty} \sup_{p^{(0)}} \| P_t - \pi \|_2^{1/2} \]

\[ \lambda(M) = 1 > \lambda_2 \ldots > \lambda_n > -1 \text{ not bipartite} \]

**Thm** Mixing Rate = \( \max \{ \lambda_2, |\lambda_n| \} \)

**Proof** Suppose \( M \) sym in \( M = M^T \) eg 6 in regular

\( q_i = (\frac{1}{n}, \ldots, \frac{1}{n}) \)

\[ P^{(0)} = \alpha \pi + P^{(1)} \quad \pi \bot P^{(0)} \]

\[ \alpha = \frac{P^{(0)^T} \pi}{\pi^T \pi} = \frac{\frac{1}{n}}{\frac{1}{n}} = 1 \]
\[ P^{(0)} = \Pi + P^{(0)'} \]

Hence

\[ \| P^{(t)} - \Pi \|_2 = \| M^T P^{(0)} - M^T \Pi \|_2 \]

\[ = \| M^T (P^{(0)} - \Pi) \|_2 \]

\[ = \| M^T P^{(0)'} \|_2 \leq \lambda^t \| P^{(0)} \| \leq \lambda^t \]