Spectral
9/17/09

Random Walk & mixing rates

$G = (V, E, w)$ $A_{ij} = w_{ij}$ (symmetric)

$d_i = \sum_j w_{ij}$ $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$

Let $p^{(t)} = \begin{pmatrix} p_1^{(t)} \\ \vdots \\ p_n^{(t)} \end{pmatrix}$ s.t. $p_i^{(t)} \equiv$ prob at v_i at time t.

<u>Claim</u> $A^T D^{-1} p^{(t)} = p^{(t+1)}$ since $A = A^T$

$M = AD^{-1}$ is the transition matrix

2 - Questions

1) $\exists$ dist $\bar{P}$ s.t. $M\bar{P} = \bar{P}$ (stationary dist)

1') $\exists!$ " "

2) $\forall P_0 \ \lim_{k \to \infty} (AD^{-1}) P_0 = \bar{P}$

1) Yes Let $d = \sum d_i$ $\quad P = \begin{pmatrix} d_1/d \\ \vdots \\ d_n/d \end{pmatrix} = \frac{1}{d} D \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

pf

$$AD^{-1}P = AD^{-1}\frac{1}{d}D\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \frac{1}{d}A\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \frac{1}{d}\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = P$$

note

$\bar{d} = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$ is an eigenvector with value 1.

1') <u>no</u> if G is not connected

2) no if G is bipartite e.g. $G =$  $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$ eigenvalue $= -1$

Note

Perron-Frobenius Thm

Suppose $A^{n \times n} \geq 0$ Graph(A) is strongly connected

<u>Def</u> $z \in \mathbb{C}$ $|z| = \sqrt{z^* z} = \sqrt{a^2 + b^2}$ $z = a + ib$

Spectral radius $\rho(A) = \max\limits_{\lambda \in \lambda(A)} |\lambda|$

<u>Thm</u> 1) $\rho(A)$ is a simple eigenvalue of A.

If x is an eigenvector for ρ then $\text{sign}(x_i) = \text{sign}(x_j)$ $\forall i, j$

2) $\theta \in \lambda(A)$ and $|\theta| = \rho(A)$ then $\theta/\rho(A)$ is an mth root of unit and all cycles in X have length a multiple of m.

3) Only non-neg eigenvector is X.

proof (to come)

eg $G =$
$$\begin{tikzcd} v_1 \arrow[r] & v_2 \arrow[r] & v_3 \arrow[ll, bend right] \end{tikzcd}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

w = 3rd root of unity

$\lambda(A) = \{w, w^2, 1\}$

$$A\begin{pmatrix} 1 \\ w \\ w^2 \end{pmatrix} = \begin{pmatrix} w \\ w^2 \\ 1 \end{pmatrix} = w\begin{pmatrix} 1 \\ w \\ w^2 \end{pmatrix}$$

If G sym then an edge is a 2-cycle

$\therefore m = \{1,2\}$ $\quad m = 2$ iff G is bipartite

For us $G = M = AD^{-1}$ not sym

Consider $N = D^{-1/2} A D^{-1/2}$ sym

Claim $\lambda(AD^{-1}) = \lambda(D^{-1/2} A D^{-1/2})$

Change of variables $q = D^{-1/2} p$

ie $D^{-1/2} A D^{-1/2} q = \lambda q$ iff $A D^{-1/2} p = \lambda p$

$$(\Leftarrow)\quad D^{-1/2} A D^{-1/2} q = D^{-1/2} A D^{-1/2} D^{-1/2} p$$

$$= D^{-1/2} A D^{-1} p$$

$$= \lambda D^{-1/2} p = \lambda q$$

$(\Rightarrow)$ Same

Find eigenvalues of N

$$N \begin{pmatrix} \sqrt{d_1} \\ \vdots \\ \sqrt{d_n} \end{pmatrix} = D^{-1/2} A D^{-1/2} D^{1/2} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = D^{-1/2} \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} \sqrt{d_1} \\ \vdots \\ \sqrt{d_n} \end{pmatrix}$$

By 3) of PF $\begin{pmatrix} \sqrt{d_1} \\ \vdots \\ \sqrt{d_n} \end{pmatrix}$ is the PF eigenvector

$$1 = \lambda_1 > \lambda_2 \cdots \geq \lambda_n \geq -1$$

G is not bipartite then $\lambda_n > -1$

By Spectral Thm

$$N = \sum_{k=1}^{n} \lambda_k v_k v_k^T$$

$$M^t = D^{1/2} N^t D^{-1/2} = \sum_{k=1}^{n} \lambda_k^t \left(D^{1/2} v_k v_k^T D^{-1/2} \right)$$

$$= Q + \sum_{k=2}^{n} \lambda_k^t \left(D^{1/2} v_k v_k^T D^{-1/2} \right)$$

where

$$Q = \frac{1}{d} \begin{pmatrix} \sqrt{d_1} & & \\ & \ddots & \\ & & \sqrt{d_n} \end{pmatrix} \begin{pmatrix} \sqrt{d_1} \\ \vdots \\ \sqrt{d_n} \end{pmatrix} \left(\sqrt{d_1} \cdots \sqrt{d_n} \right) \begin{pmatrix} 1/\sqrt{d_1} & & \\ & \ddots & \\ & & 1/\sqrt{d_n} \end{pmatrix}$$

$$= \frac{1}{d} \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \left(1 \cdots 1 \right)$$

$$= \frac{1}{d} \begin{pmatrix} d_1 & \cdots & d_1 \\ \vdots & & \vdots \\ d_n & \cdots & d_n \end{pmatrix}$$

Note $p_{ij}^{(t)} = (M^t)_{ji}$

$Q_{ji} = d_j$

$$\left(\lambda_k^t D^{1/2} V_k V_k^T D^{-1/2}\right)_{ji} = \lambda_k^t \begin{pmatrix} \sqrt{d_1} & & \\ & \ddots & \\ & & \sqrt{d_n} \end{pmatrix} \Big(\qquad \Big) \begin{pmatrix} 1/\sqrt{d_1} & & \\ & \ddots & \\ & & 1/\sqrt{d_n} \end{pmatrix}_{ji}$$

$$= \lambda_k^t \sqrt{d_j}\, (V_k)_i \cdot (V_k)_j \sqrt{d_i}^{\,-1}$$

$$= \lambda_k^t (V_k)_i (V_k)_j \left(\frac{\sqrt{d_j}}{\sqrt{d_i}}\right)$$

$$p_{ij}^t = d_j/d + \sum_{k=2}^{n} \lambda_k^t (V_k)_i (V_k)_j \frac{\sqrt{d_i}}{\sqrt{d_j}}$$

Thm $p_{ij}^t \to d_j/d \quad (t \to \infty)$

G is not bipartite

Mixing Rate

Def Mixing rate $= \mu = \limsup_{t \to \infty} \max_{i,j} \left| P_{ij}^{(t)} - \pi(j) \right|^{1/t}$

Background $f(t) \to 0$ as $t \to \infty$ $\quad f(t) \geq 0$

ie $f_1(t) = 1/t \quad f_2(t) = 1/t^2 \quad f_3(t) = 1/2^t$

$$\mu(f) = \limsup_{t \to \infty} f(t)^{1/t}$$

$$(f_1(t))^{1/t} = (1/t)^{1/t} \qquad \mu = 1$$

$$(f_3(t))^{1/t} = \left(1/2^t\right)^{1/t} = 1/2 \qquad \mu = 1/2$$

$$f_4(t) = \left(\frac{n-1}{n}\right)^t \Rightarrow \mu = \left(\frac{n-1}{n}\right)$$

Claim $\left(\frac{n-1}{n}\right)^n < 1/e$

$\left(\frac{0}{1}\right)^1 = 0 \quad \left(\frac{2-1}{2}\right)^2 = 1/4 \quad \left(\frac{3-1}{3}\right)^3 = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

$n=1$

Thm $\lim_{n \to \infty} \left(\frac{n-1}{n}\right)^n = 1/e$

Estimating Mixing Rate

1) Coupling
2) Conductance $\Rightarrow$ Spectra
3) Congestion Dilation $\Rightarrow$ Spectra
