Spectral Thm & Graph Laplacians

Spectral
9/10/09

$G = (V, E, c)$ edge weights

Def Laplacian of G, $\mathcal{L}(G) = L$

$L = D - A \qquad D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \qquad A_{ij} = c_{ij}$

Thm $L = \Gamma^T C \Gamma \qquad \Gamma =$ edge-vertex matrix

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Properties of symmetric matrices $M^T = M$

$MX = \lambda X$ for $X \neq 0$

$X \equiv$ eigenvector $\quad \lambda \equiv$ eigenvalue

$(M - \lambda I) X = 0 \quad \lambda$ st $(M - \lambda I)$ singular

ie $\det(M - \lambda I) = 0$

$\det(M - \lambda I) \equiv$ polynomial in λ of degree n.

Note: roots may be complex!

$\lambda = a + ib \quad i^2 = -1 \ \& \ a, b \in \mathbb{R}$

Complex conjugate: $(a+ib)^* = a - ib$

$A^*: \ (A^*)_{ij} = a_{ij}^*$

$A^H: \ (A^*)^T$

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1) $z \in \mathbb{C}$ then $z^* z \in \mathbb{R}$ 2) $z \in \mathbb{R}$ iff $z^* = z$

$A^H A$ is real

Def A is Hermitian if $A^H = A$

A is real symmetric $\Rightarrow$ Hermitian

Assume $A^H = A$

Fact 1 $x^H A x$ real

pf $(x^H A x)^H = x^H A^H x^{HH} = x^H A x$

Fact 2 $Ax = \lambda x \quad x \neq 0 \Rightarrow \lambda \in \mathbb{R}$

$$Ax = \lambda x$$

$$x^H A x = \lambda x^H x \qquad x \neq 0$$

$$\lambda = \frac{x^T A x}{x^T x} = \frac{\text{real}}{\text{real}} = \text{real}$$

Fact 3 $Ax = \lambda x$ & $Ay = \lambda' y$ $\lambda \neq \lambda'$ then $x^H y = 0$

$$y^H \underbrace{Ax} = y^H \lambda x = \lambda y^H x$$

$$\underbrace{y^H A} x = (Ay)^H x = \lambda' y^H x$$

$$\Rightarrow y^H x = 0$$

Cor A is real sym then eigen values & vectors are real!

Assume A is real sym

Fact 1 eigenvalues & vectors are real

Fact 2 $Ax=\lambda x$, $Ay=\mu y$, & $\lambda \neq \mu$ then $x \perp y$

Fact 3 $\exists$ orthogonal (orthonormal) eigenvectors $V_1 \cdots V_n$

pf (see notes)

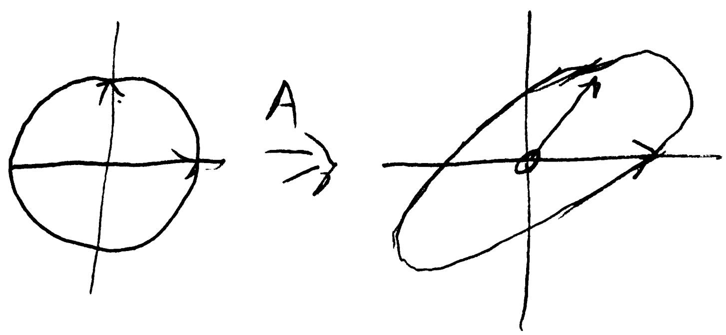
Cor $\exists$ eigenvectors $V_1 \cdots V_n$ that are indep.

Non sym example

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 = 0$$

$$\lambda = 1$$

eigenvectors x $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x = 0 \Rightarrow \langle \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \rangle$



Fact 4 $A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_n^T & - \end{pmatrix}$

Fact 5 $A = \sum_{i=1}^{n} \lambda_i (v_i v_i^T)$

Assume $L = L(G)$

Fact 1 Eigens are real

Fact 2 Eigenvalues ≥ 0

$L = \Gamma^T C \Gamma = \bar{\Gamma}^T \bar{\Gamma} \quad \bar{\Gamma} = C^{1/2}\Gamma$

$Lx = \lambda x \Rightarrow \lambda = \frac{x^T L x}{x^T x} = \frac{(\bar{\Gamma}x)^T(\bar{\Gamma}x)}{x^T x} \geq 0$

Fact 3 $Lx = 0$ iff x is constant per connected component.

$x^T L x = 0 \qquad x^T \Gamma^T C \Gamma x = \sum_{(i,j)E} c_{ij} (x_i - x_j)^2 = 0$

$x_i = x_j$ for $(i,j) \in E$.

PROOF OF SPECTRAL THEOREM

Theorem 1 (Spectral Theorem). *Let $A \in \mathbb{R}^{n\times n}$ be a real symmetric matrix. Then*

(1) *All eigenvalues of A are real.*
(2) *There exists an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^T$.*

Proof. We have proved (1) in the class. Only need to prove (2). We make an induction on n.

When $n = 1$, the claim is obvious. Now assume that the claim is valid for $n = m$, that is, for any $m\times m$-real symmetric matrix A, there exists an orthogonal matrix Q and diagonal matrix Λ such that $A = Q\Lambda Q^T$. Let us consider $(m+1)\times(m+1)$-real symmetric matrix A. By (1), A has a real eigenvalue λ with eigenvector α. We see that all entries of α must be real numbers. By Gram-Schmidt process, we may assume that there exists an *orthonormal* basis $q_1, \ldots, q_n$ with $q_1 = \alpha$. Let $P := (q_1 q_2 \cdots q_n)$ and $C := P^T AP = (c_{ij})_{(m+1)\times(m+1)}$. We claim that $c_{11} = \lambda$ and $c_{i1} = 0$ for $i \neq 1$. In fact, note that P is an orthogonal matrix, we have $AP = PC$, that is, $A(q_1 q_2 \cdots q_n) = (q_1 q_2 \cdots q_n)C$. Therefore, we have $Aq_1 = \sum_{i=1}^{m+1} c_{i1} q_i$. But $q_1 = \alpha$ is an eigenvector,so $\lambda q_1 = \sum_{i=1}^{m+1} c_{i1} q_i$. Since $q_1, \cdots, q_n$ are linearly independent. So $c_{11} = \lambda$ and $c_{i1} = 0$ for $i \neq 1$. So C has four blocks like $\begin{pmatrix} \lambda & \star \\ 0 & \tilde{A} \end{pmatrix}$. Note that $C = P^T AP$ is symmetric(why ?), thus $\star = 0$. So $C = \begin{pmatrix} \lambda & 0 \\ 0 & \tilde{A} \end{pmatrix}$ and $\tilde{A}$ has to be symmetric matrix with the size $m\times m$. By induction, there exists an orthogonal matrix Q and diagonal matrix Λ such that $\tilde{A} = Q\Lambda Q^T$. Therefore

$$C = \begin{pmatrix} \lambda & 0 \\ 0 & \tilde{A} \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & Q\Lambda Q^T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \Lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}^T$$

Therefore

$$A = PCP^T = P \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \Lambda \end{pmatrix} (P \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix})^T$$

and we easily check $P \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}$ is an orthogonal matrix and we are done. $\square$