Random Walks on Graphs

Spectral
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Graph $G = (V, E, w)$ (possibly directed)

$w : E \to \mathbb{R}^+$

$w_i = w(v_i) = \sum_{(i,j) \in E} w_{ij}$ $\quad P_{ij} = w_{ij}/w_i$

Random walk on G

Suppose, at a given time, we are at $v_i \in V$.

We move to v_j with probability P_{ij}

EG $V \equiv$ all permutations of a deck of cards

$P_{ij} \equiv$ prop of going from perm$_i$ to perm$_j$ in one shuffle.

? Why do professional players play from a deck after 5 shuffles?

EG. $V \equiv$ All web pages

$P_{ij} \equiv$ prob of going from V_i to V_j in one click

? How does (prob of being at V_i) compare to (prob of being at V_j)

Questions:

1) Does $\exists$ a steady state
2) How do we compute the prob of being at some node?
3) When are we close to the steady state

Important Parameters

<u>Access time</u> or Hitting time

$H_{ij} \equiv$ Expected time to visit j starting at i

<u>Commute Time</u>

$$K(i,j) = H(i,j) + H(j,i)$$

<u>Cover Time</u>

Expected time to visit all nodes
max over all starting nodes

<u>Mixing Rate</u> (Todo)

Random walks – the Symmetric Case

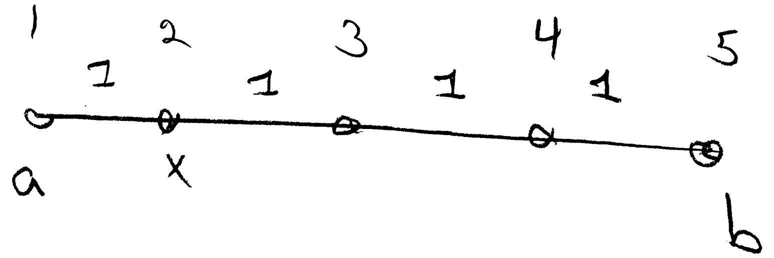
Do a random walk on a network of conductors!

Input: $G = (V, E, C)$ $\quad C_{ij} = C_{ji}$ $\quad a, b \in V$

Consider a random walk starting at x and ending at b.

<u>Def</u> $h_x \equiv$ prob we visit a before visiting b. $a \neq b$ starting from x.

EG

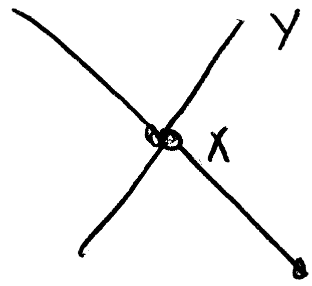


$h_a = 1 \quad h_b = 0$

h_2? $\qquad h_2 > \frac{1}{2}$ why?

$h_a = 1$ & $h_b = 0$

Suppose $x \neq a, b$



Claim $h_x = \sum_y P_{xy} h_y$

$P_{xy} \geq 0 \quad \sum_y P_{xy} = 1$

h_x is a convex combination of it neighbors!

h is harmonic with bdary a, b!

Lets solve the electrical prob

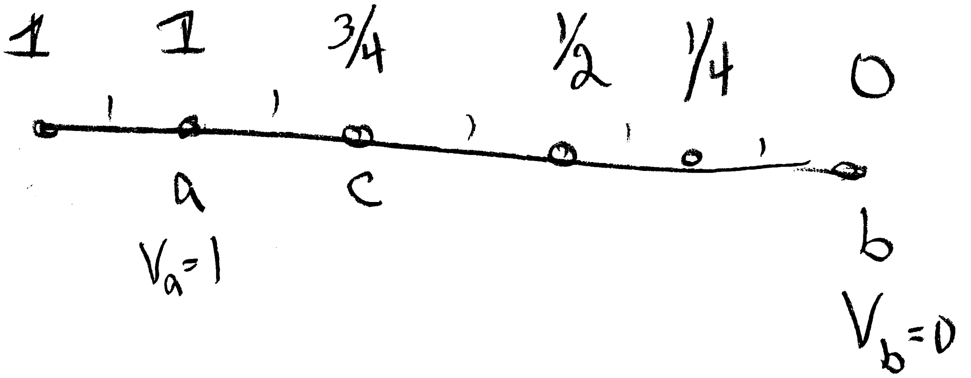
$V_a = 1$ & $V_b = 0$ and $x \neq a, b$ float.

$x \neq a, b \quad V_x = \sum_y \frac{C_{xy}}{C_x} V_y$ but $\frac{C_{xy}}{C_x} = P_{xy}$!

$\Rightarrow h = V$

Thm $V_a = 1$ & $V_b = 0$ then V_x = prob visit a before b.

EG

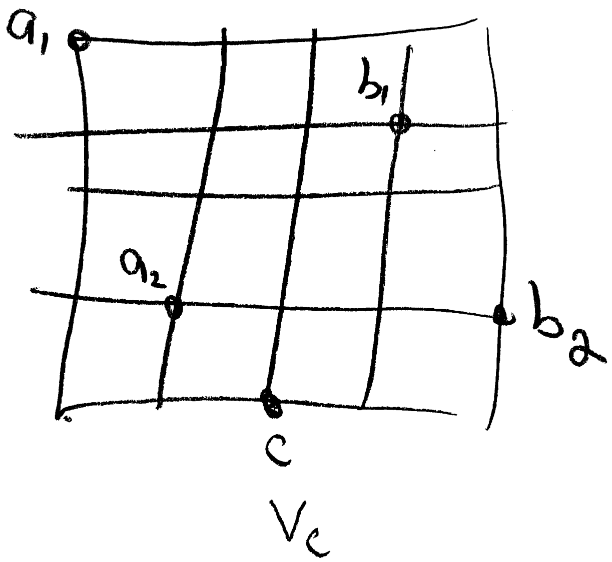


$h_c = 3/4$

What does it mean (in random walks)

If we set $V_{a_1} = V_{a_2} = 1$ & $V_{b_1} = V_{b_2} = 0$

$x \neq a_1, a_2, b_1, b_2$ float?



?

Interpretation of Current

Assume $G = (V, E, C)$ $a, b \in V$

Consider 1 unit of current flow from a to b,
Say i

What does i_{xy} correspond to in random walks?

Thm i_{xy} = Expected net # of traversals of E_{xy}
in random walk from a to b.

pf Slides 7, 8, 8A

Lets start with:

$u_x \equiv$ Expected number of visits to x before reaching b starting at a.

$u_b = 0 \qquad x \neq b$

$$u_x = \sum_y u_y P_{yx} \quad \text{note} \quad \sum_y P_{yx} \neq 1$$

Recall $C_x = \sum_y C_{xy}$

note $C_x P_{xy} = C_x \left(\frac{C_{xy}}{C_x}\right) = C_{xy} = C_{yx} = C_y \left(\frac{C_{yx}}{C_y}\right) = C_y P_{yx}$

$$u_x = \sum_y u_y \frac{C_y P_{yx}}{C_y} = \sum_y u_y \left(\frac{P_{xy} C_x}{C_y}\right)$$

$$\frac{u_x}{C_x} = \sum_y P_{xy} \left(\frac{u_y}{C_y}\right)$$

$u_x P_{xy}$ = expected # of traversals from x to y

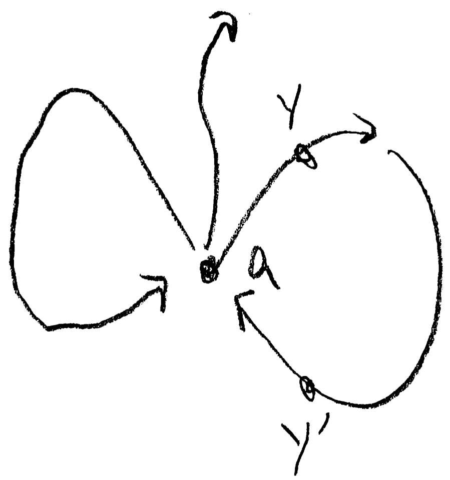
$u_y P_{yx}$ = " " y to x

j_{xy} = expect # net xy traversals.

What is net current flow from a to b?

ie $\sum_y j_{ay}$

This must be 1



This proves Thm

Let $V_x = \left(\frac{u_x}{C_x}\right)$ then

$$V_x = \sum_y P_{xy} V_y \quad V_x \text{ is harmonic!}$$

What is the bdary! $V_b = 0$

Suppose we knew u_a $\quad V_a = u_a / C_a$

$\therefore$ V is a voltage where $V_b = 0$ & $V_a = u_a/C_a$

Let i_{xy} be its current

$$i_{xy} = (V_x - V_y)\, C_{xy} = \left(\frac{u_x}{C_x} - \frac{u_y}{C_y}\right) C_{xy}$$

$$= u_x\left(\frac{C_{xy}}{C_x}\right) - u_y\left(\frac{C_{yx}}{C_y}\right) = u_x P_{xy} - u_y P_{yx}$$

How to compute hitting Time

<u>Def</u> $h(x,b)$ = expected time to reach b from x

$h_x = h(x,b)$ b fixed

Lets write a recurrance

$$h_b = 0 \qquad x \neq b$$

$$h_x = 1 + \sum_y h_y P_{xy} \qquad (*)$$

How do we solve $(*)$?

Lets think of h_x as a voltage V_x

$$V_b = 0 \qquad V_x = 1 + \sum_y \frac{C_{xy}}{C_x} V_y$$

$$C_x V_x = C_x + \sum_y C_{xy} V_y$$

$$\underbrace{C_x V_x - \sum_y C_{xy} V_y}_{\text{Graph Laplacian}} = \underbrace{C_x}_{\text{residual current}}$$

$$\underset{V_n = 0}{LV} = \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \\ \delta \end{pmatrix} \qquad \begin{array}{l} c = \sum c_i \\ b = V_n \end{array}$$

by conservation of flow

$$\delta = c_n - c$$

<u>Alg</u> for hitting time

solve $\underset{V_n = 0}{LV} = \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \\ c_n - c \end{pmatrix}$

return V_x

What about commute time?

$a = v_1$ & $b = v_n$

Solution 1

solve $LV^b = \begin{pmatrix} c_n \\ \vdots \\ c_n - c \end{pmatrix}$ $\quad LV^a = \begin{pmatrix} c_1 - c \\ \vdots \\ c_n \end{pmatrix}$

$$h(1,n) = V_1^b - V_n^b$$

$$h(n,1) = V_n^a - V_1^a \qquad V = V^b - V^a$$

$$C(1,n) = (V^b - V^a)_1 - (V^b - V^a)_n$$

Solution 2

$$L(V^b - V^a) = LV^b - LV^a = \begin{pmatrix} c_1 \\ \vdots \\ c_n - c \end{pmatrix} - \begin{pmatrix} c_1 - c \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \\ -c \end{pmatrix} = c\begin{pmatrix} 1 \\ 0 \\ \vdots \\ -1 \end{pmatrix}$$

solve $LV = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ -1 \end{pmatrix}$

return $C(V_1 - V_n)$ but $(V_1 - V_n) = R_{1n}$

<u>Thm</u> $C(a,b) = R_{ab} \cdot C$