Resistive Model of a Graph & Random Walks

Motivation: Making a recommendation (NETFLIX)

Question: Should we recommend M to V?

Score(V, M)

Idea 1

Score(V, M) = graph dist from V to M

\[ W_{ij} = \frac{1}{\text{rank} \, k_i} \]

\[ \text{Score}(V, M) = \min_{VP_{iM}} W(P) \]

Idea 2

\[ W(P) = \min_{E \in P} (\text{rank}(E)) \]

\[ \text{Score}(V, M) = \max_{VP_{iM}} W(P) \]
Problem: For 1) and 2) extra paths do not improve score.

Idea 3: \[ \text{Score}(V, M) = \text{max flow from } V \text{ to } M. \]

Problem: Shorter paths do not improve score.

Idea: View edges as conductors.

\[ \text{Score}(v, M) = \text{effective conductance} \]
Resistance Theory

Ohms Law:

\[ V = I \times R \]

\[ C = \text{conductance} \]
\[ R = \text{resistance} \]
\[ V = \text{voltage} \]
\[ I = \text{current} \]

\[ C = \frac{1}{R} \quad I = C \times V = \frac{V}{R} \]

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Facts no proof  Resistors in series

\[ V_0, R_1, V_1, R_2 \quad \ldots \quad R_m, V_m \]

\[ R = R_1 + \ldots + R_m \]
\[ C = \frac{1}{\left( \frac{1}{C_1} + \ldots + \frac{1}{C_m} \right)} = ? \]

i.e. \[ I = \frac{V}{R} \]
Conductors in Parallel

\[ C = C_1 + \ldots + C_m \]

\( i \equiv i = V \cdot C \)

Effective Resistance/Conductance

Let \( G \) be a network of resistors

\[ \text{Def} \quad R_{ab} = \frac{V_{ab}}{i_{ab}} \quad C_{ab} = \frac{1}{R_{ab}} \]
HW: Show that $R_{ab}$ is a metric space

i.e. 1) $R_{ab} \geq 0$
2) $R_{ab} = 0$ iff $a = b$
3) $R_{ab} = R_{ba}$
4) $R_{ac} \leq R_{ab} + R_{bc}$
Computing effective resistance

Use Kirchhoff's Law in flow in = flow out

An example

\[ V_1 - C_1(V - V_1) \]
\[ V_2 - C_2(V - V_2) \]
\[ V_3 - C_3(V - V_3) \]

Residual current \( i_1 + i_2 + i_3 \)

By Kirchhoff

\[ i_1 + i_2 + i_3 = 0 \]

\[ C_1(V - V_1) + C_2(V - V_2) + C_3(V - V_3) = 0 \]

\[ (C_1 + C_2 + C_3)V = C_1V_1 + C_2V_2 + C_3V_3 \]
\[ C = C_1 + C_2 + C_3 \]
\[ CV = C_1 V_1 + C_2 V_2 + C_3 V_3 \]
\[ V = \frac{C_1}{C} V_1 + \frac{C_2}{C} V_2 + \frac{C_3}{C} V_3 \]

*V* is convex combination of \( V_1, V_2, V_3 \)

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**residual current** = \( CV - C_1 V_1 - C_2 V_2 - C_3 V_3 \)

The general case

\[ G = (V, E, C) \quad C : E \rightarrow \mathbb{R}^+ \]

\[ V = \{ V_1, \ldots, V_n \} \]

\[ d(V_i) = \sum_{(i,j) \in E} C_{ij} \]

\[ A_{ij} = \begin{cases} C_{ij} & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases} \]
\[ \text{Laplacian}(G) = L(G) = L \]

\[ L_{ij} = \begin{cases} 
  d(V_i) & \text{if } i = j \\
  -C_{ij} & \text{if } (i, j) \in E \\
  0 & \text{otherwise} 
\end{cases} \]

i.e. \[ L = D - A \quad \text{where} \quad D = \begin{pmatrix} d(V_1) & 0 & \cdots & 0 \\
 0 & \ddots & \ddots & \vdots \\
 0 & \cdots & d(V_{n-1}) & 0 \\
 0 & \cdots & 0 & d(V_n) \end{pmatrix} \]

Let \( V \) be a voltage setting of nodes

Note \( (LV)_i \) = residual current at \( V_i \)

Inverse: We inject current and get voltages.

The net injected must be zero!
Goal: $R_{in}$

method 1 solve $L \begin{pmatrix} 0 \\ V_n \\ \vdots \\ V_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -i \end{pmatrix} \Rightarrow \begin{pmatrix} \hat{c} \\ \hat{V} \end{pmatrix} = \begin{pmatrix} V/R \\ 1 \end{pmatrix}$

$R = V_i$

$\begin{pmatrix} \hat{c} \\ \hat{V} \end{pmatrix}$ is called a boundary valued prob.

In our case $V_i \& V_n$ are the bndary

$(V_1, \ldots, V_n)$ is called harmonic

because $V_i$ is interior $\Rightarrow$

$V_i$ is convex combination of neighbors
Maximum Principle  If $f$ is harmonic then min & max are on bdary

If $f$ is interior then \exists \text{eig} \ V_i \ & \ V_j \ s.t.

$V_i \leq V \leq V_j$

Uniqueness Principle  If $f \ & \ g$ are harmonic with same bdary values then $f = g$

$\text{Pt} \ f-g$ is harmonic with zero on bdary

$\Rightarrow \ f-g = 0 \Rightarrow f = g$
method 2 solve $LV = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ Does $V$ exist?

$$R_{in} = V_i - V_n$$

Another way to view the Laplacian

Edge-vertex Matrix

Orient each edge

$$\Gamma = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$
Let \( c_1, \ldots, c_m \equiv \text{conductance of } e_1, \ldots, e_m \)

\[
C = \begin{pmatrix}
  c_1 & 0 \\
  0 & c_m
\end{pmatrix}
\]

**Note:** \( \Gamma V \equiv \text{voltage drop across each edge} \)
\( \Sigma \Gamma V \equiv \text{current flow } \)" \( \Gamma^T C \Gamma V \equiv \text{residual current at each vertex} \)

**Thus** \( L = \Gamma^T C \Gamma \)
Current & Energy/Power Dissipation

\[ \frac{C}{R} \]

\[ \begin{array}{c}
\text{Newton} \\
\text{Energy} = \text{Force} \times \text{Distance} \\
= \text{Volt} \times \text{Current} \\
= V \times i \\
= CV^2 \\
= i^2 R
\end{array} \]

\[ i = CV \]
\[ i_R = V \]

Network

\[ E = \frac{1}{2} \sum_{x,y} i_{xy} (V_x - V_y) \]

\[ V^T L V = V^T \Gamma^T C \Gamma V = (\Gamma V)^T C (\Gamma V) \]
\[ = \sum_{\text{Oriented}} C_{xy} (V_x - V_y)^2 = E \]
\[ (x,y) \in E \]
Suppose \( a, b \in V \) effective resistance \( R_{ab} \)

Effective energy \( i_{ab}^2 R_{ab} = R_{ab} \) if \( i_{ab} = 1 \)

Real energy using Kirchhoff’s Law

Solve \( LV = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \)
\( V_a = v_1 \quad V_b = v_2 \)

\[ \text{Energy} = v^T L v = v^T \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = V_a - V_b \]

Thus, real Kirchhoff energy \( \approx \) effective energy

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Minimum Energy Flow

**Def (OR type flow)**

\( j : E \rightarrow R \) is a flow from \( a \) to \( b \) if

1) \( j_{xy} = -j_{yx} \)

2) \( \sum_{Y \in \mathcal{Y}} j_{xy} = 0 \) if \( x \neq a, b \)

3) \( j_{xy} = 0 \) \( (x,y) \notin E \)
Def. \( \dot{j}_x = \sum_{Y} \dot{j}_{xy} \) (residual flow at \( x \))

Let \( W \) = any voltage settings
\( \dot{j} \) = any flow from \( a \) to \( b \)

Conservation of Energy
\[
(W_a - W_b) \dot{j}_a = \frac{1}{2} \sum_{X,Y} (W_x - W_y) \dot{j}_{xy}
\]

\[
\text{L.H.S.} = \sum_{X,Y} (W_x - W_y) \dot{j}_{xy} = \sum_{X} W_x \sum_{Y} \dot{j}_{xy} - \sum_{Y} W_y \sum_{X} \dot{j}_{xy}
\]
\[
= W_a \sum_{Y} \dot{j}_{ay} + W_b \sum_{Y} \dot{j}_{by} - (W_a \sum_{X} \dot{j}_{xa} + W_b \sum_{Y} \dot{j}_{xb})
\]
\[
= W_a \ddot{j}_a + W_b \ddot{j}_b - W_a (-\ddot{j}_a) - W_b (-\ddot{j}_b)
\]
\[
= W_a \ddot{j}_a - W_b \ddot{j}_a + W_a \dot{j}_a - W_b \dot{j}_a = 2(W_a - W_b) \ddot{j}_a
\]
Thomson's Principle

i is a unit Kirchhoff flow from a to b
j is any unit flow from a to b

Then \( \sum i_{xy}^2 R_{xy} \leq \sum j_{xy}^2 R_{xy} \)

Proof: Let \( d = j - i \) then \( d \) is a zero flow in \( d_a = 0 \)

\[
\sum j_{xy}^2 R_{xy} = \sum (i_{xy} + d_{xy})^2 R_{xy}
\]

\[
= \sum i_{xy}^2 R_{xy} + 2 \sum i_{xy} R_{xy} d_{xy} + \sum d_{xy}^2 R_{xy}
\]

\[
+ 2 \sum (v_x - v_y) d_{xy} \quad (\star)
\]

Set \( W = V \) & \( d = 0 \) then by conservation of energy

\( (\star) \equiv 4 (v_a - v_b) d_a = 0 \) thus

\[
\sum j_{xy}^2 R_{xy} = \sum i_{xy}^2 R_{xy} + \sum d_{xy}^2 R_{xy}
\]

\[
\geq \sum i_{xy}^2 R_{xy}
\]
Rayleigh's Monotonicity Law

If $A_{xy} \bar{R}_{xy} \geq R_{xy}$ then $\overline{E_{R_{ab}}} \geq E_{R_{ab}}$

Let $i = \text{unit flow from a to b in } \bar{R}$

$$i = \bar{R}$$

$$\overline{E_{R_{ab}}} = \int \overline{E_{R_{ab}}} = \frac{1}{2} \sum i_{xy}^2 \bar{R}_{xy}$$

$$\geq \frac{1}{2} \sum i_{xy}^2 R_{xy}$$

$$\geq \frac{1}{2} \sum i_{xy}^2 R_{xy} \quad (\text{Thomson})$$

$$= \overline{E_{R_{ab}}}$$