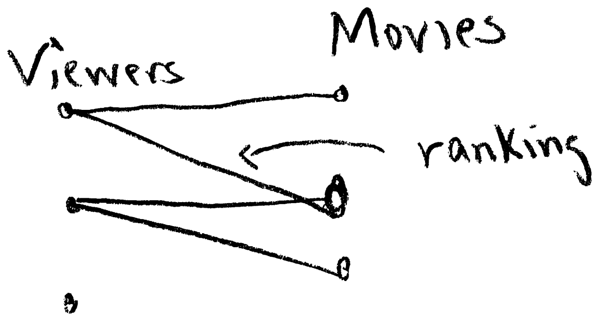
Spectral 9/8/09

Resistive Model of a Graph & Random Walks

Modivation: Making a recommendation (NETFLIX)



Question: Should we recommend M to V?

Score(V, M)

Idea 1 Score(V,M) = graph dist from V to M

$$W_{ij} = \frac{1}{\text{rank}_{ij}}$$

$$\text{Score}(V,M) = \min_{VPM} W(P)$$

Idea 2 $W(P) = \min_{e \in P}(\text{rank}(e))$

$$\text{Score}(V,M) \equiv \max_{VPM} W(P)$$

Problem For 1) and 2) extra paths do not improve score

Idea 3 $\text{Score}(V, M) \equiv$ Max flow from V to M.

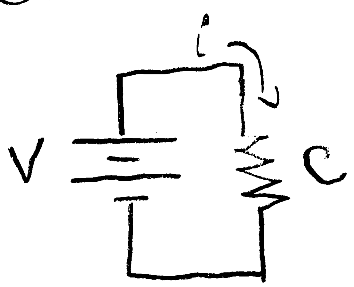
Problem Shorter paths do not improve score

Idea View edges as conductors

$\text{Score}(V, M) =$ effective conductance

Resistance Theory

Ohms Law:



$C \equiv$ conductance

$R \equiv$ resistance

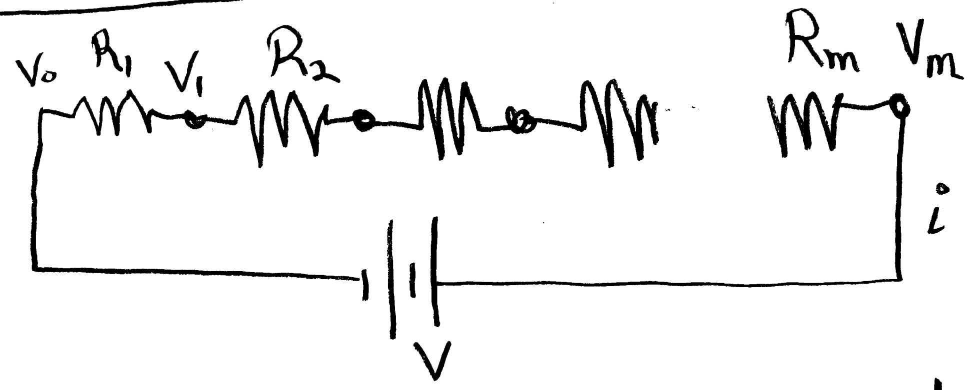
$V \equiv$ voltage

$i \equiv$ current

$C = 1/R \qquad i = C \cdot V = V/R$

Facts no proof

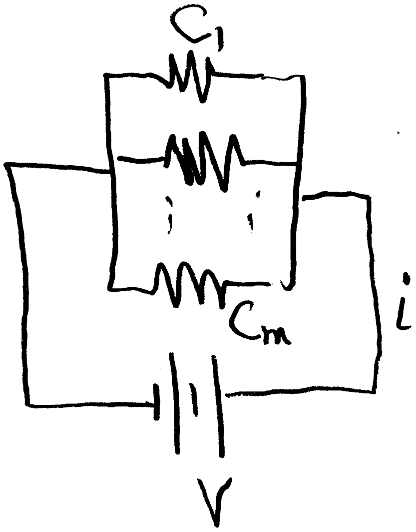
Resistors in series



$R = R_1 + \cdots + R_m \qquad C = \dfrac{1}{(1/C_1 + \cdots 1/C_m)} = ?$

i.e. $i = V/R$

Conductors in Parallel

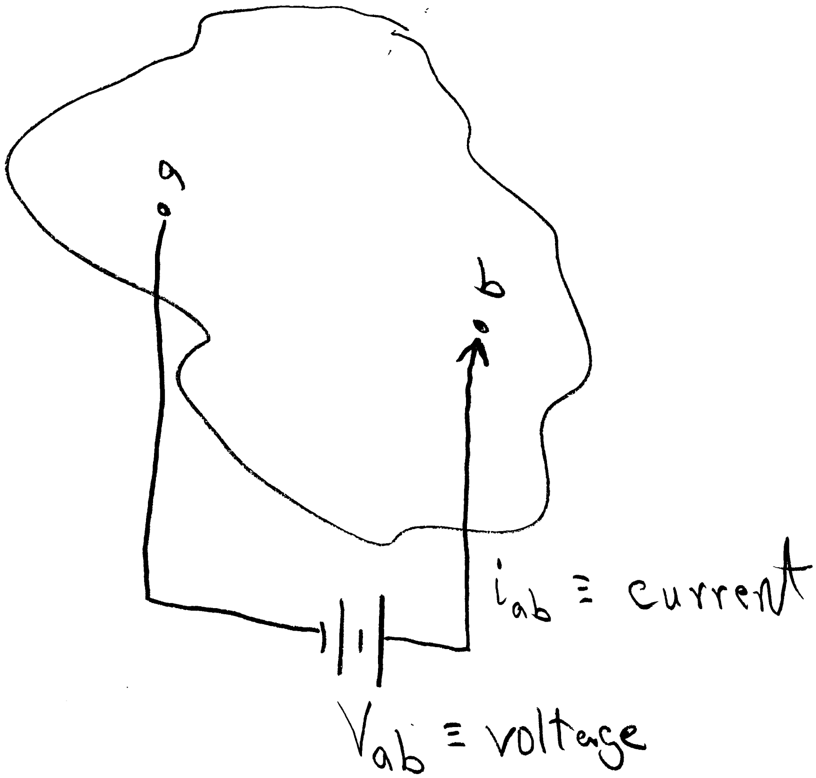


$$C = C_1 + \cdots + C_m$$

ie $i = V \cdot C$

Effective Resistance/Conductance

Let G be a network of resistors



Def $R_{ab} = V_{ab}/i_{ab}$ $\quad C_{ab} = 1/R_{ab}$

HW) Show that R_{ab} is a metric space

ie 1) $R_{ab} \geq 0$

2) $R_{ab} = 0$ iff $a = b$

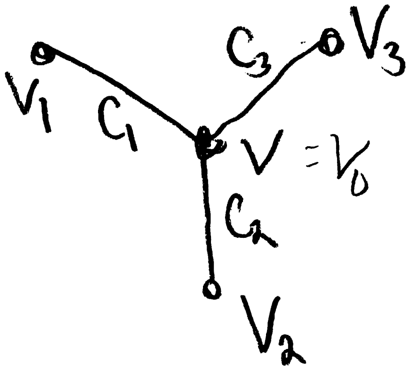
3) $R_{ab} = R_{ba}$

4) $R_{ac} \leq R_{ab} + R_{bc}$

Computing effective resistance

Use Kirchhoff's Law ie flow in = flow out

an example



by Ohm's Law

$$i_1 = C_1(V - V_1)$$
$$i_2 = C_2(V - V_2)$$
$$i_3 = C_3(V - V_3)$$

residual current $\equiv i_1 + i_2 + i_3$

by Kirchhoff

$$i_1 + i_2 + i_3 = 0$$

$$C_1(V - V_1) + C_2(V - V_2) + C_3(V - V_3) = 0$$

$$(C_1 + C_2 + C_3)V = C_1V_1 + C_2V_2 + C_3V_3$$

$$C = C_1 + C_2 + C_3$$

$$CV = C_1 V_1 + C_2 V_2 + C_3 V_3$$

$$V = \frac{C_1}{C} V_1 + \frac{C_2}{C} V_2 + \frac{C_3}{C} V_3$$

V is convex combination of $V_1, V_2 V_3$

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residual current $= CV - C_1 V_1 - C_2 V_2 - C_3 V_3$

The general case

$$G = (V, E, C) \quad C : E \to R^+$$

$$V = \{v_1, \dots, v_n\}$$

$$d(v_i) = \sum_{(i,j) \in E} c_{ij}$$

$$A_{ij} = \begin{cases} c_{ij} & \text{if } (i,j) \in E \\ 0 & \text{o.w} \end{cases}$$

$$\text{Laplacian}(G) = L(G) = L$$

$$L_{ij} = \begin{cases} d(v_i) & \text{if } i=j \\ -C_{ij} & \text{if } (i,j) \in E \\ 0 & \text{o.w} \end{cases}$$

ie $L = D - A$ where $D = \begin{pmatrix} d(v_1) & & 0 \\ & \ddots & \\ 0 & & d(v_n) \end{pmatrix}$

Let V be a voltage setting of nodes

Note $(LV)_i$ = residual current at V_i

Inverse: We inject current and get voltages.

The net injected must be zero!

Goal: R_{in}

method 1 solve $L\begin{pmatrix} 0 \\ V_2 \\ \vdots \\ V_{n-1} \\ 1 \end{pmatrix} \overset{!}{=} \begin{pmatrix} i \\ 0 \\ \vdots \\ 0 \\ -i \end{pmatrix}$ (*)

$i = V/R$

$V = 1$

$R = 1/i$

return $1/i$

(*) is called a boundary valued prob.

In our case V_1 & V_n are the bdary

$(V_1, \dots, V_n)$ is called harmonic

because $V_i \in$ interior $\Rightarrow$

V_i is convex combination of neighbors

<u>Maximum Principle</u> If f is harmonic then min & max are on bdary

<u>pf</u> $v \in$ interior then $\exists$ neig v_i & v_j st

$$v_i \leq v \leq v_j$$

<u>Uniqueness Principle</u> If f & g are harmonic with same bdary values then $f = g$

<u>pf</u> $f - g$ is harmonic with zero on bdary

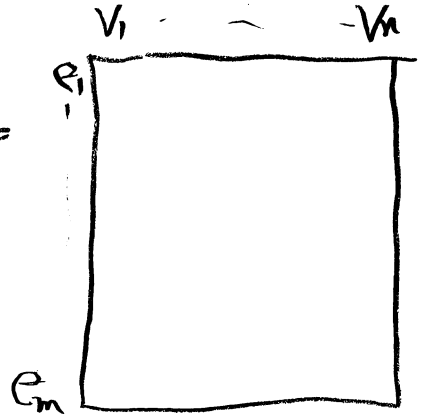
$$\Rightarrow f - g \equiv 0 \Rightarrow f = g$$

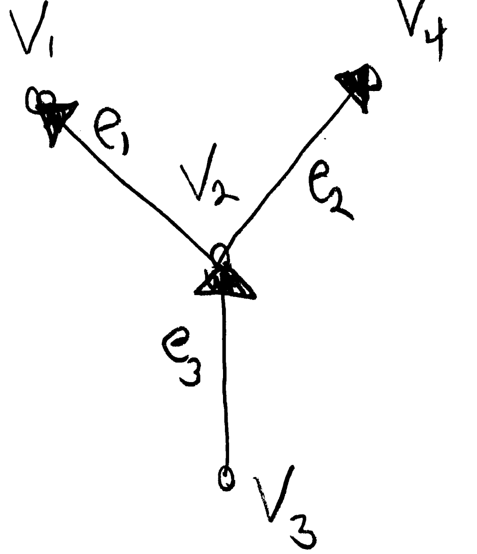
<u>method 2</u> solve $LV = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}$ Does V exist?

$R_{1n} = V_1 - V_n$

Another way to view the Laplacian

Edge-vertex Matrix

$\Gamma^{m \times n} =$ 



Orient each edge

$$\Gamma = \begin{array}{c|cccc} & V_1 & V_2 & V_3 & V_4 \\ \hline e_1 & 1 & -1 & 0 & 0 \\ e_2 & 0 & -1 & 0 & 1 \\ e_3 & 0 & 1 & -1 & 0 \end{array}$$

Let $c_1 \cdots c_m \equiv$ conductance of $e_1 \cdots e_m$

$$C = \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_m \end{pmatrix}$$

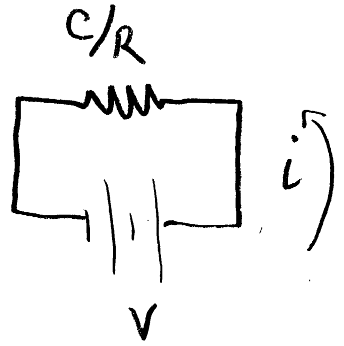
<u>Note</u> $\Gamma V \equiv$ voltage drop across each edge

$C\Gamma V \equiv$ current flow " "

$\Gamma^T C \Gamma V \equiv$ residual current at each vertex

<u>Thus</u> $L = \Gamma^T C \Gamma$

Current & Energy/Power Dissipation



<u>Newton</u> Energy $\equiv$ Force $\cdot$ Distance

$\equiv$ Volt $\cdot$ current

$\equiv V \cdot i$

$i = CV$ $\qquad \equiv CV^2$

$iR = V$ $\qquad \equiv i^2 R$

Network $\qquad E = \frac{1}{2} \sum_{x,y} i_{xy} (V_x - V_y)$

$$V^T L V = V^T \Gamma^T C \Gamma X = (\Gamma V)^T C (\Gamma V)$$

$$= \sum_{\substack{\text{Oriented} \\ (x,y) \in E}} C_{xy} (V_x - V_y)^2 = E$$

Suppose $a, b \in V$ effective resistance R_{ab}

Effective energy $i_{ab}^2 R_{ab} = R_{ab}$ if $i_{ab} = 1$

Real energy using Kirchhoff's Law

Solve $LV = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ -1 \end{pmatrix}$ $V_a = V_1$ $V_b = V_n$

$$\text{Energy} \equiv V^T L V = V^T \begin{pmatrix} 1 \\ 0 \\ \vdots \\ -1 \end{pmatrix} = V_a - V_b$$

<u>Thm</u> Real Kirchhoff energy $\equiv$ effective energy

Minimum Energy flow

<u>Def</u> (OR type flow)

$j : E \to \mathbb{R}$ is a flow from a to b if:

1) $j_{xy} = -j_{yx}$

2) $\sum_y j_{xy} = 0$ if $x \neq a, b$

3) $j_{xy} = 0$ $(x,y) \notin E$

Def $\dot{j}_x = \sum_y \dot{j}_{xy}$ (residual flow at x)

Let $W \equiv$ any voltage settings
$\dot{j} \equiv$ any flow from a to b

<u>Conservation of Energy</u>

$$(W_a - W_b)\,\dot{j}_a = \tfrac{1}{2} \sum_{x,y} (W_x - W_y)\,\dot{j}_{xy}$$

pf

$$2\text{RHS} = \sum_{x,y} (W_x - W_y)\,\dot{j}_{xy} \equiv \sum_x W_x \sum_y \dot{j}_{xy} - \sum_y W_y \sum_x \dot{j}_{xy}$$

$$= W_a \sum_y \dot{j}_{ay} + W_b \sum_y \dot{j}_{by}$$

$$- \left(W_a \sum_x \dot{j}_{xa} + W_b \sum_y \dot{j}_{xb} \right)$$

$$= W_a \dot{j}_a + W_b \dot{j}_b - W_a(-\dot{j}_a) - W_b(-\dot{j}_b)$$

$$= W_a \dot{j}_a - W_b \dot{j}_a + W_a \dot{j}_a - W_b \dot{j}_a = 2(W_a - W_b)\,\dot{j}_a$$

Thomson's Principle

i is a unit Kirchhoff flow from a to b

j is any unit flow from a to b

then $\sum i_{xy}^2 R_{xy} \le \sum j_{xy}^2 R_{xy}$

Pf Let $d = j - i$ Then d is a zero flow ie $d_a = 0$

$$\sum j_{xy}^2 R_{xy} = \sum (i_{xy} + d_{xy})^2 R_{xy}$$

$$= \sum i_{xy}^2 R_{xy} + 2 \sum i_{xy} R_{xy} d_{xy} + \sum d_{xy}^2 R_{xy}$$

$$+ 2 \sum (V_x - V_y) d_{xy} \quad (*)$$

Set $W = V$ & $j = d$ then by conseration of energy

$(*) \equiv 4 (V_a - V_b) d_a = 0$ thus

$$\sum j_{xy}^2 R_{xy} = \sum i_{xy}^2 R_{xy} + \sum d_{xy}^2 R_{xy}$$

$$\ge \sum i_{xy}^2 R_{xy}$$

Rayleigh's Monotonicity Law

If $\forall_{xy}\ \bar{R}_{xy} \geq R_{xy}$ then $\overline{ER}_{ab} \geq ER_{ab}$

Pf Let $j \equiv$ unit flow from a to b in $\bar{R}$

$i \equiv$ " " R

$$\overline{ER}_{ab} = 1^2\, \overline{ER}_{ab} = \frac{1}{2} \sum j_{xy}^2 \bar{R}_{xy}$$

$$\geq \frac{1}{2} \sum j_{xy}^2 R_{xy}$$

$$\geq \frac{1}{2} \sum i_{xy}^2 R_{xy} \quad \text{(Thomson)}$$

$$= ER_{ab}$$