Spectral Graph Theory and its Applications	September 16, 2004
Lecture 5	
Lecturer: Daniel A. Spielman	

5.1 Introduction

In this lecture, we will prove the following theorem:

Theorem 5.1.1. *Let G be a planar graph with n vertices of maximum degree d, and let λ_2 be the second-smallest eigenvalue of its Laplacian. Then,*

$$\lambda_2 \leq \frac{8d}{n}.$$

The proof will involve almost no calculation, but will use some special properties of planar graphs. However, this proof has been generalized to many planar-like graphs, including the graphs of well-shaped 3d meshes. Recently, Jon Kelner has generalized it graphs of bounded genus.

I begin by recalling two definitions of planar graphs.

Definition 5.1.2. *A graph is planar if there exists an embedding of the vertices in $\mathbb{R}^2$, $f : V \to \mathbb{R}^2$ and a mapping of edges $e \in E$ to simple curves in $\mathbb{R}^2$, $f_e : [0,1] \to \mathbb{R}^2$ such that the endpoints of the curves are the vertices at the endpoints of the edge, and no two curve intersect in their interiors.*

Definition 5.1.3. *A graph is planar if there exists an embedding of the vertices in $\mathbb{R}^2$, $f : V \to \mathbb{R}^2$ such that for all pairs of edges (a,b) and (c,d) in E, with a, b, c, and d distinct, the line segment from $f(a)$ to $f(b)$ does not cross the line segment from $f(c)$ to $f(d)$.*

These definitions are equivalent.

5.2 Geometric Embeddings

We typically upper bound λ_2 by evidencing a test vector. Here, we will upper bound λ_2 by evidencing a test embedding. The bound we apply is:

Lemma 5.2.1. *For any $d \geq 1$,*

$$\lambda_2 = \min_{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n \in \mathbb{R}^d : \sum \boldsymbol{v}_i = \boldsymbol{0}} \frac{\sum_{(i,j)\in E} \|\boldsymbol{v}_i - \boldsymbol{v}_j\|^2}{\sum_i \|\boldsymbol{v}_i\|^2}. \tag{5.1}$$

Proof. Let $\boldsymbol{v}_i = (x_i, y_i, \ldots, z_i)$. We note that

$$\sum_{(i,j)\in E} \|\boldsymbol{v}_i - \boldsymbol{v}_j\|^2 = \sum_{(i,j)\in E} (x_i - x_j)^2 + \sum_{(i,j)\in E} (y_i - y_j)^2 + \cdots + \sum_{(i,j)\in E} (z_i - z_j)^2.$$

Similarly,

$$\sum_i \|\boldsymbol{v}_i\|^2 = \sum_i x_i^2 + \sum_i y_i^2 + \cdots + \sum_i z_i^2.$$

It is now trivial to show that $\lambda_2 \geq RHS$: just let $x_i = y_i = \cdots = z_i$ be given by an eigenvector of λ_2. To show that $\lambda_2 \leq RHS$, we apply my favorite inequality: $\frac{A+B+\cdots+C}{A'+B'+\cdots+C'} \geq \min\left(\frac{A}{A'}, \frac{B}{B'}, \ldots, \frac{C}{C'}\right)$, and then recall that $\sum x_i = 0$ implies

$$\frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{\sum_i x_i^2} \geq \lambda_2.$$

□

For an example, consider the natural embedding of the square with corners $(\pm 1, \pm 1)$.

The key to applying this embedding lemma is to obtain the right embedding of a planar graph. Usually, the right embedding of a planar graph is given by Koebe's embedding theorem, which I will now explain. I begin by considering one way of generating planar graphs. Consider a set of circles $\{C_1, \ldots, C_n\}$ in the plane such that no pair of circles intersects in their interiors. Associate a vertex with each circle, and create an edge between each pair of circles that meet at a boundary. The resulting graph is clearly planar. Koebe's embedding theorem says that *every planar graph results from such an embedding.*

Theorem 5.2.2 (Koebe). *Let $G = (V, E)$ be a planar graph. Then there exists a set of circles $\{C_1, \ldots, C_n\}$ in $\mathbb{R}^2$ that are interior-disjoint such that circle C_i touches circle C_j if and only if $(i, j) \in E$.*

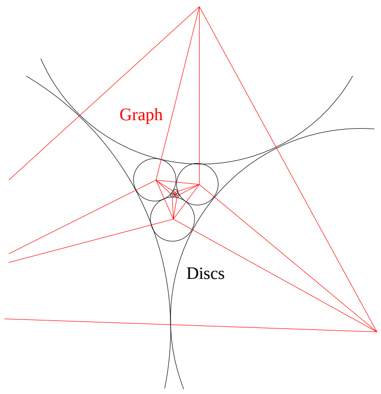
This is an amazing theorem, which I won't prove today. You can find a proof at

`http://math.mit.edu/~spielman/course/lect3.html`

Such an embedding is often called a *kissing disk* embedding of the graph. From a kissing disk embedding, we obtain a natural choice of $\boldsymbol{v}_i$: the center of disk C_i. Let r_i denote the radius of this disk. We now have an easy upper bound on the numerator of (5.1): $\|\boldsymbol{v}_i - \boldsymbol{v}_j\|^2 = (r_i + r_j)^2 \leq 2r_i^2 + 2r_j^2$. On the other hand, it is trickier to obtain a lower bound on $\sum \|\boldsymbol{v}_i\|^2$. In fact, there are graphs whose kissing disk embeddings result in

$$(5.1) = \Theta(1).$$

These graphs come from triangles inside triangles inside triangles... Such a graph is depicted below:



We will fix this problem by lifting the planar embeddings to the sphere by stereographic projection. Given a plane, $\mathbb{R}^2$, and a sphere S tangent to the plane, we can define the stereographic projection map, Π, from the plane to the sphere as follows: let $\boldsymbol{s}$ denote the point where the sphere touches the plane, and let $\boldsymbol{n}$ denote the opposite point on the sphere. For any point $\boldsymbol{x}$ on the plane, consider the line from $\boldsymbol{x}$ to $\boldsymbol{n}$. It will intersect the sphere somewhere. We let this point of intersection be $\Pi(\boldsymbol{x})$.

The fundamental fact that we will exploit about stereographic projection is that *it maps circles to circles!* So, by applying stereographic projection to a kissing disk embedding of a graph in the plane, we obtain a kissing disk embedding of that graph on the sphere. Let $D_i = \Pi(C_i)$ denote the image of circle C_i on the sphere. We will now let $\boldsymbol{v}_i$ denote the center of D_i, on the sphere.

If we had $\sum_i \boldsymbol{v}_i = \boldsymbol{0}$, the rest of the computation would be easy. For each i, $\|\boldsymbol{v}_i\| = 1$, so the denominator of (5.1) is n. Let r_i denote the straight-line distance from $\boldsymbol{v}_i$ to the boundary of D_i. We then have

$$\|\boldsymbol{v}_i - \boldsymbol{v}_j\|^2 \leq (r_i + r_j)^2 \leq 2r_i^2 + 2r_j^2.$$

So, the denominator of (5.1) is at most $2d \sum_i r_i^2$. On the other hand, the area of the cap encircled by D_i is at least πr_i^2. As the caps are disjoint, we have

$$\sum_i \pi r_i^2 \sum 4\pi,$$

which implies that the denominator of (5.1) is at most

$$2d \sum_i r_i^2 \leq 8d.$$

Putting these inequalities together, we see that

$$\min_{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n \in \mathbb{R}^d : \sum \boldsymbol{v}_i = \boldsymbol{0}} \frac{\sum_{(i,j)\in E} \|\boldsymbol{v}_i - \boldsymbol{v}_j\|^2}{\sum_i \|\boldsymbol{v}_i\|^2 .} \leq \frac{8d}{n}.$$

Thus, we merely need to verify that we can ensure that

$$\sum_i \boldsymbol{v}_i = \boldsymbol{0}. \tag{5.2}$$

Note that there is enough freedom in our construction to believe that we could prove such a thing: we can put the sphere anywhere on the plane, and we could even scale the image in the plane before placing the sphere. By carefully combining these two operations, it is clear that we can place the center of gravity of the $\boldsymbol{v}_i$s close to any point on the boundary of the sphere. It turns out that this is sufficient to prove that we can place it at the origin.

5.3 The center of gravity

We need a nice family of maps that transform our kissing disk embedding on the sphere. It is particularly convenient to parameterize these by a point ω inside the sphere. For any point α on the surface of the unit sphere, I will let Π_α denote the stereographic projection from the plane tangent to the sphere at α. I will also define Π_α^{-1}. To handle the point $-\alpha$, I let $\Pi_\alpha^{-1}(-\alpha) = \infty$, and $\Pi_\alpha(\infty) = -\alpha$. We also define the map that dilates the plane tangent to the sphere at α by a factor a: D_α^a. We then define

$$f_\omega(\boldsymbol{x}) \stackrel{\text{def}}{=} \Pi_{\omega/\|\omega\|}\left(D_{\omega/\|\omega\|}^{1-\|\omega\|}\left(\Pi_{\omega/\|\omega\|}^{-1}\right)\right).$$

For $\alpha \in S$ and $\omega = a\alpha$, this map pushes everything on the sphere to a point close to α. As a approaches 1, the mass gets pushed closer and closer to α.

Instead of proving that we can achieve (5.2), I will prove a slightly simpler theorem. The proof of the theorem we really want is similar, but about just a few minutes too long for class. We will prove

Theorem 5.3.1. *Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be points on the unit-sphere. Then, there exists a circle-preserving map from the unit-sphere to itself.*

The reason that this theorem is different from the one that we want to prove is that if we apply a circle-preserving map from the sphere to itself, the center of the circle might not map to the center of the image circle.

To show that we can achieve $\sum_i \boldsymbol{v}_i = \boldsymbol{0}$, we will use the following topological lemma, which follows immediately from Brouwer's fixed point theorem. In the following, we let B denote the ball of points of norm less than 1, and the the sphere of points of norm 1.

Lemma 5.3.2. *If $\phi : B \to B$ be a continuous map that is the identity on S. Then, there exists an $\omega \in B$ such that*

$$\phi(\omega) = \boldsymbol{0}.$$

We will prove this lemma using Brouwer's fixed point theorem:

Theorem 5.3.3 (Brouwer). *If $g : B \to B$ is continuous, then there exists an $\alpha \in B$ such that $g(\alpha) = \alpha$.*

Proof of Lemma 5.3.2. Let b be the map that sends $\boldsymbol{z} \in B$ to $\boldsymbol{z}/\|\boldsymbol{z}\|$. The map b is continuous at every point other than $\boldsymbol{0}$. Now, assume by way of contradiction that $\boldsymbol{0}$ is not in the image of ϕ, and let $g(\boldsymbol{z}) = -b(\phi(\boldsymbol{z}))$. By our assumption, g is continuous and maps B to B. However, it is clear that g has no fixed point, contradiction Brouwer's fixed point theorem. □

Lemma 5.3.2, was our motivation for defining the maps f_ω in terms of $\omega \in B$. Now consider setting

$$\phi(\omega) = \frac{1}{n} \sum_i f_\omega(\boldsymbol{v}_i).$$

The only thing that stops us from applying Lemma 5.3.2 at this point is that ϕ is not defined on S, because f_ω was not defined for $\omega \in S$. To fix this, we define for $\alpha \in S$

$$f_\alpha(\boldsymbol{z}) = \begin{cases} \alpha & \text{if } \boldsymbol{z} \neq -\alpha \\ -\alpha & \text{otherwise.} \end{cases}$$

We then encounter the problem that $f_\alpha(\boldsymbol{z})$ is not a continuous function of α. To fix this, we set

$$h_\omega(\boldsymbol{z}) = \begin{cases} 1 & \text{if } \operatorname{dist}(\omega, \boldsymbol{z}) < 2 - \epsilon, \text{ and} \\ (2 - \operatorname{dist}(\omega, \boldsymbol{z}))/\epsilon & \text{otherwise.} \end{cases}$$

Now, the function $f_\alpha(\boldsymbol{z}) h_\alpha(\boldsymbol{z})$ is continuous, because $h_\alpha(-\alpha) = 0$. So, we may set

$$\phi(\omega) \stackrel{\text{def}}{=} \frac{\sum_i f_\omega(\boldsymbol{v}_i) h_\omega(\boldsymbol{v}_i)}{\sum_i h_{\omega/\|\omega\|}(\boldsymbol{v}_i),}$$

which is now continuous and is the identity map on S.

So, for any $\epsilon > 0$, we may now apply Lemma 5.3.2 to find an ω for which

$$\phi(\omega) = \mathbf{0}.$$

It is a simple exercise to verify that for ϵ sufficiently small, the ω we find will have norm bounded away from 1, and so $h_\omega(\boldsymbol{v}_i) = 1$ for all i, in which case

$$\sum_i f_\omega(\boldsymbol{v}_i) = \mathbf{0},$$

as desired.