Spectral Graph Theory and its Applications	September 14, 2004

Lecture 4

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4.1 Corrections to Problem Set

1.c Prove that there exists a $c \in \{1, -1\}$ such that $v_i(j) = cw_i(\pi(j))$, rather than $v_i(j) = w_i(\pi(j))$.

4. Prove that
$$\lambda_2(P) \geq \min_{i:v_i \neq 0} \frac{(L_P v)_i}{v_i}.$$

4.2 Cutting Graphs

One of the inspirations for spectral graph theory came from the problem of graph partitioning. In a graph partitioning problem, one tries to partition a graph into roughly equally sized pieces while minimizing the number of edges between those pieces. Graph partitioning is a fundamental primitive in many algorithms. For example, when a graph represents the communication required in a computation, graph partitioning is used to partition the problem among parallel processors. Graph partitioning is also used as a primitive in many divide-and-conquer algorithms.

There are many variants of the graph partitioning problem. For now, we will concentrate on partitions into two pieces. Partitions into many pieces can be obtained by applying such partitions recursively.

One's first instinct is often to try to divide a graph into equal sized pieces while minimizing the number of edges cut. But, this might not provide the best cuts. For example, consider a graph composed of a clique on $2n/3$ vertices a clique on $n/3$ vertices, with one edge between them. If we cut the edge between the cliques, the graph will no longer be balanced. But, this cut seems better than any balanced cut.

So, one is often willing to sacrifice balance for cutting fewer edges. A commonly used tradeoff between balance and number of edges cut is given by the *cut ratio* ϕ. If $S, \bar{S}$ is a partition of the vertices of a graph, then
$$\phi(S) = \frac{e(S, \bar{S})}{\min\left(|S|, |\bar{S}|\right)},$$
where $e(S, \bar{S})$ denotes the number of edges between S and $\bar{S}$. This measure has the nice property that if S_1 and S_2 are disjoint and $|S_1 \cup S_2| \leq n/2$, then
$$\phi(S_1 \cup S_2) \leq \max\left(\phi(S_1), \phi(S_2)\right).$$

The cut of minimum ratio is given by the set S minimizing $\phi(S)$. The quality of this cut is called the `isoperimetric number` of the graph, and is given by

$$\phi(G) \stackrel{\text{def}}{=} \min_{S \subset V} \phi(S).$$

4.3 An integer program

Donnath and Hoffman (check it was them) proposed approximating $\phi(G)$ be the following integer program:

$$\min_{x \in \{0,1\}^n} \frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{\sum_{i<j}(x_i - x_j)^2}. \tag{4.1}$$

If we set x to be the characteristic vector of a set S, then the numerator is $e(S, \bar{S})$ and the denominator is $|S|\left|\bar{S}\right|$. So, the value of (4.1) lies between $\phi(G)/n$ and $\phi(G)/(n/2)$. Thus, if we could solve (4.1), we could approximate the isoperimetric number of a graph to within a factor of 2.

However, there is no easy way to solve this integer program. So, we will relax the program to obtain one that we can solve. Instead of restricting x to be 0 or 1, we could let each x_i lie in the interval $[0, 1]$. As we will see momentarily, we can solve the resulting program. We should note that making this relaxation allows us to obtain a lower minimum. However, it is not clear how much lower it can be. It is also not clear whether we can use the vector x we obtain to find a good partition.

To see that we can solve this program, first note that multplying x by any constant does not change the value of the ratio. So, we may as well let $x - I$ lie in $[0, \infty)$. Second, replacing x by $x + c$ for any $c \in \mathbb{R}$ leaves both the numerator and denominator unchanged. So, we can let x be any vector in $\mathbb{R}^n$. Without loss of generality, we can choose x so that $\sum_i x_i = 0$. We can then simplify the denominator by observing

$$\begin{aligned}
2\sum_{i<j}(x_i - x_j)^2 &= \sum_i \sum_{j \neq i}(x_i^2 - 2x_i x_j + x_j^2) \\
&= 2(n-1)\left(\sum_i x_i^2\right) - 2\sum_i x_i \sum_{j\neq i} x_j \\
&= 2(n-1)\left(\sum_i x_i^2\right) + 2\sum_i x_i^2 \\
&= 2n \sum_i x_i^2.
\end{aligned}$$

So, we find that the value of the relaxed program is λ_2/n:

$$\min_{x \in \{0,1\}^n} \frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{\sum_{i<j}(x_i - x_j)^2}. \geq \min_{x \in \mathbb{R}^n} \frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{\sum_{i<j}(x_i - x_j)^2} = \min_{x \in \mathbb{R}^n, x \perp \mathbf{1}} \frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{n\sum_i x_i^2} = \lambda_2/n.$$

Thus, we find that

$$\phi(G) \geq \lambda_2/2.$$

4.4 Cheeger's Inequality

We now address the question of how well $\lambda_2/2$ approximates $\phi(G)$, and whether or not we can find a good partition from v_2, the corresponding eigenvector. The first question is addressed by a discrete version of Cheeger's inequality, proved by Jerrum and Sinclair (and Alon, and Diaconis and Strook?):

$$\lambda_2 \geq \frac{\phi^2}{2d_{max}},$$

where d_{max} is the maximum degree of a vertex in the graph. The proof of the discrete Cheeger inequality also yields a partition.

However, exactly computing eigenvectors is a pain. So, we will prove a theorem of Mihail that finds a cut from any test vector. In particular, the cut will consist of all vertices i such that $x_i < t$ for some threshold t.

Theorem 4.4.1. *Let $x \perp \mathbf{1}$ and assume, without loss of generality, that $x_1 \leq x_2 \leq \cdots \leq x_n$. Then, for some i,*

$$\frac{x^T L x}{x^T x} \geq \frac{\phi\left(\{1, \ldots, i\}\right)^2}{2d_{max}},$$

where d_{max} is the maximum degree of a vertex in the graph.

A proof of this theorem may be found at

`http://www-math.mit.edu/~spielman/AEC/lect5.ps`